Modular transformations of admissible $N = 2$ and Affine sl(2|1;C) characters

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Modular Transformations of Admissible $N = 2$ and Affine $\widehat{sl}(2|1;\mathbb{C})$ Characters

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A thesis presented for the degree of
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Centre for Particle Theory
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Dedicated to

My Wife

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Abstract

This thesis is a study of the affine super-algebra $\widehat{sl}(2|1;\mathbb{C})$ and $N = 2$ superconformal algebra at fractional levels.

In the first chapter we review background material on Conformal Field Theory, and how it appears in the context of string theory and the Wess - Zumino - Novikov - Witten model. We also discuss integrable and admissible representations of infinite dimensional algebras and their modular transformations.

In Chapter 2 we elaborate some more on modular transformations and we derive them in the case of non - unitary minimal $N = 2$ characters. Some very explicit formulas are presented.

In Chapter 3 we discuss character formulas for the affine $\widehat{sl}(2|1;\mathbb{C})$ algebra and some of their general properties are given, in particular their behaviour under spectral flow.

In Chapter 4 we turn to the study of sumrules for $\widehat{sl}(2|1;\mathbb{C})$ at level k. These involve the product of $\widehat{sl}(2)$ characters at level k, k', and 1 with $(k+1)(k'+1) = 1$. We consider $k + 1 = \frac{p}{u}$ for $(p, u) = 1$, $p \in \mathbb{Z}^*$, $u \in \mathbb{N}$ and show that the sumrules we have obtained agree with the literature when the parameter p is restricted to $p = 1$. We use the integral form of the sumrules to study the modular properties of $\widehat{sl}(2|1)$ characters at fractional level in the last section of Chapter 4.

The advisor for this work has been Dr. Anne Taormina.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, the University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction and Background

Twentieth - century physics has witnessed the triumph of symmetry and its precise formulation in theoretical language. The work of Lie and Cartan paved the way for the general application of symmetries in microscopic physics within quantum mechanics. Wigner, probably the most important figure in the application of group theory to physics, fitted the possible elementary particles into representations of the Lorentz and Poincare groups. The principles of special and general relativity - the seeds of the other great revolution of twentieth - century physics - were also motivated by the appeal of symmetry. Modern theories of elementary particles (the so - called standard model) rest on the principle of local gauge symmetry. Our understanding of phase transitions and critical phenomena draws a great deal on the concept of broken symmetry. In particular broken gauge symmetries are central to our understanding of weak interactions, superconductivity and cosmology. So symmetry is a very powerful organising principle in physics and one can witness many symmetries in nature.

Two - dimensional conformal symmetry has been an important tool in theoretical physics during the last decade. Its origins can be traced back on the one hand to statistical mechanics, and on the other hand to string theory. Historically the most important impetus came from statistical mechanics, where it described and classified critical phenomena. Mainly after 1984 the subject went through a period of rapid development because of its importance for string theory. In addition there has been important input from mathematics, in particular through the work of Kac and collaborators. One can distinguish yet another separate origin of some ideas, namely from work on rigorous approaches to quantum field theory. Another reason of this interest lies certainly in the beautiful mathematical structure of Quantum Field Theory (QFT).

In this chapter we start by reviewing conformal symmetry, string theory and the Wess - Zumino - Novikov - Witten (WZNW) model. Finally we will show some mathematical aspects of this subject related to N = 2 and $s\hat{l}(2/1)$ super-algebras.

1.1 Conformal Transformations

In this section, we briefly review the basic properties of Conformal Field Theory (CFT) in d dimensions, with particular emphasis on the case of two dimensions [1]. A conformal field theory in d - dimensions is a related quantum field theory which is invariant under conformal transformations. These form a special class of general coordinate transformations under which the metric is rescaled by a local scalar function $\Lambda(x)$:

$$\begin{aligned} x &\to x', \\ g_{\mu\nu}(x) &\to \Lambda(x) g_{\mu\nu}(x), \end{aligned} \tag{1.1}$$

where $\mu, \nu = 1, ..., d$. So, conformal transformations act on the metric as Weyl transformations (note that an arbitrary transformation $x \to x'$ has the following effect on the metric,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)). \tag{1.2}$$

They are therefore local changes of scale which preserve the angle between any two vectors u and w at a given point, $u.w = u^\mu g_{\mu\nu} w^\nu$.

We will study this conformal group first. In flat space - time of dimension (n, m) $(n + m \neq 2)$ they form a group isomorphic with $SO(n + 1, m + 1)$. However, in the complex plane it is well- known that all (anti-) analytic transformations are conformal. This extends to the Minkowski plane where in light-cone coordinates, the conformal transformations are given by:

$$x^{\pm} \to x'^{\pm}(x^{\pm}). \tag{1.3}$$

For a space of Euclidean signature, it is advantageous to use a complex basis $(\tau + i\sigma, \tau - i\sigma)$. In string theory, the space in which these coordinates live is a cylinder, as σ is used as periodic coordinate. This also applies to two - dimensional statistical systems with periodic boundary conditions in one dimension. One then maps this cylinder to the full complex plane with coordinates,

$$(z, \bar{z}) = (exp(\tau + i\sigma), exp(\tau - i\sigma)), \tag{1.4}$$

where we will take a flat metric proportional to δ_{ij} in the real coordinates, or in the complex coordinates,

$$ds^2 = 2\sqrt{g}dzd\bar{z}. \tag{1.5}$$

In complex coordinates, a conformal transformation is given by:

$$z \to z' = f(z), \qquad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}), \tag{1.6}$$

where f is an analytic function, and $\bar{f}$ is anti-analytic. A primary field transforms under a conformal transformation as:

$$\phi(z, \bar{z}) \to \phi'(f(z), \bar{f}(\bar{z})) = (\partial f(z))^{-h}(\bar{\partial}\bar{f}(\bar{z}))^{-\bar{h}}\phi(z, \bar{z}). \tag{1.7}$$

The numbers h and $\bar{h}$ are called the conformal dimensions of the field ϕ. For infinitesimal transformations of the coordinates $f(z) = z - \epsilon$, we see that the primary fields transform as :

$$\delta_\epsilon\phi(z, \bar{z}) = \epsilon(z)\partial\phi(z, \bar{z}) + h\partial\epsilon(z)\phi(z, \bar{z}). \tag{1.8}$$

By choosing for $\epsilon(z)$ any power of z we see that the conformal transformations form an infinite algebra generated by $L_m = -z^{m+1}$ and $\bar{L}_m = -\bar{z}^{m+1}$, $m \in \mathbb{Z}$.

A CFT is always characterised by its scale invariance. As in any local field theory, scale invariance implies full conformal invariance. Scale invariance is equivalent to the vanishing of the trace of energy - momentum tensor. In complex coordinates this means that $T_{z\bar{z}} = 0$. Hence there are only two independent components of energy - momentum tensors T_{zz}, $T_{\bar{z}\bar{z}}$ and their conservation law becomes:

$$\partial^z T_{zz} = 0 \qquad \partial_{\bar{z}} T_{zz} = 0$$
$$\partial^{\bar{z}} T_{\bar{z}\bar{z}} = 0 \qquad \partial_z T_{\bar{z}\bar{z}} = 0, \tag{1.9}$$

implying that $T_{zz} = T(z)(T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}))$ is a holomorphic (anti holomorphic) function of $z(\bar{z})$. We recall that if a field theory has a conserved, trace-less energy momentum tensor, it is invariant both under general coordinate transformations and Weyl transformations. The generators of infinitesimal conformal transformations are :

$$L_n = \int_C \frac{dz}{2\pi i} z^{n+1} T(z), \tag{1.10}$$

and similarly for $\bar{L}_n$. The contour circles the origin only once.

By using (1.10) we have,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}. \tag{1.11}$$

In a quantum theory $T$and $\bar{T}$ become operators, and by using the operator product expansion $T(z)T(w)$ and (1.11) we will arrive at

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}, 0, \qquad [L_n, c] = 0, \tag{1.12}$$

where c is the central charge. Clearly, L_0 corresponds to scaling transformations in z. The combination $L_0 + \bar{L}_0$ generates scaling transformations in the complex plane $x \to \lambda x$, while $i(L_0 - \bar{L}_0)$ generates rotations. This means that a field $\phi(x)$ with conformal dimensions h and $\bar{h}$ has scaling dimensions $h+\bar{h}$ and $|h-\bar{h}|$. The three generators $\{L_{-1}, L_0, L_1\}$ are associated to infinitesimal Möbius transformations. They are obtained from the vector fields $1, z, z^2$. These are the conformal Killing vectors on the sphere. The integrated form of the infinitesimal transformations is the group of fractional linear transformations $SL(2;\mathbb{C})$:

$$z' \to \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}, \qquad ad - bc = 1, \tag{1.13}$$

thus L_{-1} generates translations, L_0 generates scalar transformations, and L_1 generates special conformal transformations.

1.2 String Theory

A striking application of conformal symmetry is provided by string theory. The universe seems to contain a large number of elementary particles. It is an appealing idea to think of these particles as different states of one single object. This would enable us to treat them in a symmetric way , the simplest objects in every- day experience which have such different eigenstates are strings. We will attempt to give a brief introduction to the subject of strings to give the flavour for some important concepts which will be needed in conformal field theory. In recent years great efforts have been spent in the quest for a theory of quantum gravity, which probably must be a unifying theory of all known interactions. Based on the principles of mechanics and local gauge invariance, the standard model is a widely recognised and experimentally tested quantum field theory of electro - weak and strong interactions. Einstein's theory of general covariance and the equivalence principle, is likewise an accepted and tested classical field theory of gravity. As we know, gravity is a weak force and negligible in particle scattering experiments done in order to test the standard model. A prime candidate for a theory of quantum gravity is (super-) string theory.The terminology originates from the basic description of elementary particles as excitations of one dimensional objects [2] [3] [4] [5] [6] called strings, of the size of the Planck length.

Strings are not the subject of the present thesis but serve as a motivation for studying conformal field theory. One then has to find which action governs a string like object moving through space - time. The simplest action, was found by Polyakov . Here one considers a field theory defined on the world sheet of string. Let $x^{\mu}(\sigma, \tau)$ be D real variables that depend on two independent real variables σ and τ. The index μ will be allowed to take values $1, 2, ..., D$, the first $D - 1$ corresponding to space - like coordinates. The quantity σ will be assumed to lie in the interval $0 \leq \sigma \leq \pi$, but τ will be allowed to take any real value. The interpretation will be that each fixed value of σ specifies a particular point of the string, different values of σ corresponding to different points, the D coordinates $x^{\mu}(\sigma, \tau)$ describe the trajectory of that point in a D - dimensional space - time. The variables σ and τ will be called world sheet - coordinates, σ being described as the spatial and τ as the time coordinate, whereas the x^{μ} will be referred to as space - time coordinates. If

$$x^{\mu}(0,\tau) = x^{\mu}(\pi,\tau), \tag{1.14}$$

for $\mu = 1, 2, ..., D$ and all values of τ, the string is said to be closed string. Closed bosonic strings are described by means of the bosonic action:

$$S(X,g) = -\frac{T}{2}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\delta_\alpha X^\mu \delta_\beta X_\mu, \tag{1.15}$$

defined on a two - dimensional surface with the topology of cylinder (for the non - interacting string, at least), where g= det $g_{\alpha,\beta}$ and $\alpha, \beta = 1, 2$. The parameter T is the string tension. It is often rewritten as $T = \frac{1}{2\pi\alpha'}$, where α' has dimensions of length square.

Here $X^{\mu}(\sigma, \tau)$ is a map from two dimensional space (called the world sheet) to space time (often target space). This function defines the embedding of the string in space time, as a function of the proper time τ, i.e. specifies where a point σ along the string is located at proper time τ.

The conformal invariance of that action plays an important role in the proper quantisation of string theory in Minkowski space. If we now proceed to quantise the bosonic string theory we must consider the path integral as following,

$$Z = \int D^d X D g e^{-S(X,g)}. \tag{1.16}$$

We know that the $g_{\alpha\beta}$ is an integral over the intrinsic shapes of 2d surfaces, whereas the X^{μ} integral is over the different ways of embedding a 2d surface into D - dimensional space - time. Indeed, the path integral (1.15)also describes interacting strings with just a slight change of boundary conditions. If one requires string interactions to be local, then strings can only interact when they touch. Two strings that touch can join together, to become one, or in the time - reversed process a single string can split into two [7]. So interactions between strings involve their creation and annihilation and would most naturally be described in the framework of second quantisation. For the interaction case the partition function will be as,

$$Z = \sum_{genus\ h=0}^{\infty} (g_s)^{2h-2} \int D^d X D g_{\alpha\beta} e^{-S(X,g)}. \tag{1.17}$$

This is called the Polyakov path integral. All the surfaces are closed. The number of handles of a 2d surface characterises its genus and is denoted by h, $h = 0, 1, 2, \ldots$. The surfaces of lowest genus are the sphere ($h = 0$) and the torus ($h = 1$). The action (1.15) has a very large symmetry group, corresponding to re-parametrisation of the world - sheet, and rescaling of the metric $g^{\alpha\beta}$. These invariances are quite natural from the point of view of string theory. So when viewing the theory defined by (1.17) as a field theory in two dimensions, a first surprise awaits us. The field theory has an infinite dimensional symmetry group. A second surprise arises when we quantise the bosonic string theory. Requiring that the symmetry survives quantisation fixes the number of space - time dimensions to 26. By adding an extra action to (1.15), one can make a consistent quantum theory for other dimensions of space - time than 26, the noncritical strings. It is the two dimensional world - sheet metric that becomes a dynamical quantum field and in a certain gauge the surviving freedom is assigned to the so - called Liouville field. Within the framework of non - critical strings, much progress has been made in describing the coupling of minimal conformal matter to 2D gravity. Extending the Virasoro algebra, which is inherent in every CFT, leads to more complicated and perhaps even more realistic string theories. As it is well-known since Einstein that the metric is related to gravity, the study of consistent quantum actions for the metric provides a quantisation of gravity in two dimensions. The Polyakov action $S(x, g)$ is invariant under general coordinate transformations $x^\mu \to x'^\mu$, local Weyl rescaling of the metric $g_{\mu\nu}(x) \to \Lambda(x) g_{\mu\nu}(x)$, and of the fields $\phi(x) \to \Lambda(x)^h \phi(x)$, h being the scaling dimension of the field $\phi(x)$. We see that the group formed by the conformal transformations is infinite dimensional in two dimensions. This makes clear why the symmetry group of string theory is so exceptionally large. The conformal transformations are generated by the energy - momentum tensor T^{ij} of the theory, which has scaling dimension $h = 2$.

So in Minkowski space-time we can obtain the Virasoro generators L_n from the bosonic string theory as quantities bilinear in oscillator modes which are the Fourier modes of the solution to the equations of motion,

$$\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0, \tag{1.18}$$

1.2 String Theory

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$$X^\mu(\sigma,\tau) = x^\mu + \sqrt{\alpha'}\alpha_0^\mu\tau + \frac{i\sqrt{\alpha'}}{2}\sum_{n\in\mathbb{Z}\ 0}\left(\frac{\alpha_n^\mu}{n}e^{-i(\tau+\sigma)n} + \frac{\bar{\alpha}_n^\mu}{n}e^{-i(\tau-\sigma)n}\right). \tag{1.19}$$

That $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$ is evident.

In quantising the theory in the canonical manner, one makes the α_n^μ into operators and demands that they satisfy the following commutation relations,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad [\alpha_m^\mu, \bar{\alpha}_n^\mu] = 0. \tag{1.20}$$

Then with,

$$L_n = \sum_{m\in\mathbb{Z}} : \alpha_{n-m}^\mu \alpha_m^\mu :, \tag{1.21}$$

one can recover the Virasoro algebra from the commutation relations. In that case the central charge c will be the dimension of space-time.

1.3 Wess - Zumino - Novikov- Witten Model

In 1984, Belavin, Polyakov and Zamolodchikov [8] showed how an infinite - dimensional field theory problem could effectively be reduced to a finite problem, by the presence of an infinite - dimensional symmetry. The symmetry algebra was the Virasoro algebra, or two - dimensional symmetry, and the field theories studied were examples of two dimensional conformal field theories.. They showed how to solve the minimal models of conformal field theory, so - called because they realise just the Virasoro algebra, and they do it in a minimal fashion. All fields in these models could be grouped into a discrete, finite set of conformal families, each associated with a representation of the Virasoro algebra. This strategy has since been extended to a large class of conformal field theories with similar structure, the rational conformal field theories [9]. The new feature is that the theories realise infinite - dimensional algebras that contain the Virasoro algebra as sub-algebra. Special among these infinite - dimensional algebras are the affine Kac - Moody algebras, realised in the Wess - Zumino -Novikov - Witten model (WZNW). They are the simplest infinite - dimensional extensions of ordinary semi - simple Lie algebras, and much is known about them, and also about the WZNW model. In here we introduce the WZNW model, including its current algebra.

A Lagrangian realisation of CFT with an affine symmetry is the WZNW theory [10] [11] [12] [13] . Let's see how that model realises $\hat{g} \oplus \hat{g}$ as current algebra [10] [14] . Given a group G, we can define a corresponding σ model, which is a theory with action,

$$S_0 = \frac{k}{16\pi} \int \int d^2x Tr(\partial_a g \partial^a g^{-1}), \tag{1.22}$$

where g is a field that takes its values in the group G. This action is not conformally invariant. However, one can make the theory conformally invariant by adding a Wess - Zumino term, leading to the action,

$$S = S_0 + \frac{k}{24\pi} \int \int \int d^3y f^{abc} Tr[(g\partial_a g^{-1})(g\partial_b g^{-1})(g\partial_c g^{-1})]. \tag{1.23}$$

In here k is the level of the corresponding affine Lie algebra. In the first term g is a map from the two dimensional manifold to G. In the second term the strange feature is that the integral is over a three - dimensional volume V , so g is a map from a three dimensional manifold to G. However the integral is a total derivative, and hence it can be written as a surface integral over the boundary of V, which is the 2- surface used in the first term . The extra term is required to make the theory conformally invariant. In general then we can say that in the second term, the integral is over a 3 - dimensional space which has a physical two - dimensional space as its boundary. The above action is invariant under transformations of the groups G and $\bar{G}$ (the groups G and $\bar{G}$ are isomorphic). The separate conservation of the currents J_z and $\bar{J}_{\bar{z}}$ implies the invariance of the action under,

$$g(z, \bar{z}) \rightarrow \omega(z) g(z, \bar{z}) \bar{\omega}^{-1}(\bar{z}), \tag{1.24}$$

where ω and $\bar{\omega}$ are two arbitrary matrices valued in G and $\bar{G}$ respectively.

For infinitesimal transformations $\omega(z) = 1 + \epsilon(z)$, $\bar{\omega}(z) = 1 + \bar{\epsilon}(\bar{z})$, the WZNW field g transforms as

$$\delta_\epsilon g = \epsilon g, \qquad \delta_{\bar{\epsilon}} g = -g\bar{\epsilon}. \tag{1.25}$$

The action is invariant under transformations $\delta g = \delta_\epsilon g + \delta_{\bar{\epsilon}} g$, in other words, $\delta S = 0$ for S given in (1.23).

The equations of motion of the WZNW model are

$$\partial^\mu (g^{-1}\partial_\mu g) + i\epsilon_{\mu\nu}\partial^\mu (g^{-1}\partial^\nu g) = 0. \tag{1.26}$$

Switching to the complex coordinates z, $\bar{z}$, and using $\partial^\mu = 2\partial_z, \epsilon_{\mu\nu} = \frac{i}{2}$, these give

$$\partial_z (g^{-1}\partial_{\bar{z}} g) = 0, \tag{1.27}$$

with hermitian conjugate

$$-\partial_{\bar{z}}(\partial_z g g^{-1}) = 0. \tag{1.28}$$

Defining

$$J := -k\partial_z g g^{-1}, \qquad \bar{J} := k g^{-1}\partial_{\bar{z}} g, \tag{1.29}$$

we have

$$\partial_z J = 0, \qquad \partial_{\bar{z}} \bar{J} = 0. \tag{1.30}$$

So the currents J and $\bar{J}$ are purely holomorphic and anti-holomorphic respectively. These currents will realise two copies of the affine algebra $\hat{g}$. In the Euclidean path integral formulation, a correlation function of the product X of fields is given by

$$\langle X \rangle = \frac{\int [d\phi] X e^{-S[\phi]}}{\int [d\phi] e^{-S[\phi]}}, \tag{1.31}$$

where $[d\phi]$ indicates path integration over the fields ϕ of the theory. If the action S transforms with $\delta S = \int_c dz \delta s(z)$, then

$$\delta\langle X \rangle = -\int_c dz \langle (\delta s) X \rangle. \tag{1.32}$$

So by taking $\omega = \sum_a \omega^a t^a$, $J = \sum_a J^a t^a$ where t^a are the generators of G and using Nother's theorem in the WZNW action we can write

$$\delta_\omega \langle X \rangle = \frac{1}{2\pi i} \int_c dz \sum_a \omega^a(z) \langle J^a(z) X \rangle, \tag{1.33}$$

where we have put $\bar{\omega} = 0$ for simplicity. Putting $X = J^b(\omega)$, with $J(w) = -k(\partial_z g)g^{-1}$ and $\delta_\omega g = \omega g$, we get

$$\delta_\omega J = [\omega, J] - k\partial_\omega w. \tag{1.34}$$

More explicitly this is

$$\delta_\omega J^b(\omega) = \sum_{c,d} i f^{bcd} w^c(\omega) J^d(\omega) - k\partial_\omega \omega^b(\omega). \tag{1.35}$$

In (1.33) this gives

$$\frac{1}{2\pi i} \sum_{c,d} \int d\omega \omega^c(\omega) < i f^{cbd} \frac{J^d(\omega)}{z-\omega} + \frac{k\delta_{bc}}{(z-\omega)^2} >= \frac{1}{2\pi i} \int d\omega \omega^a(\omega) < J^a(z) J^b(\omega) > . \tag{1.36}$$

This relation determines the singular part of the product expansion (OPE) of the two currents $J^a(z)$ and $J^b(w)$,

$$J^a(z) J^b(w) = \frac{k\delta_{ab}}{(z-w)^2} + \frac{i f^{abc} J^c(w)}{z-w} + regular. \tag{1.37}$$

A similar OPE holds for the currents $\bar{J}^a(\bar{z})$. The Laurent expansion of currents about $z = 0$ is $J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$, or equivalently, $J^a_n = \frac{1}{2\pi i} \int_c dz z^n J^a(z)$.

We can translate this expansion, so that

$$J^a(z) = \sum_{n \in \mathbb{Z}} (z-w)^{-1-n} J^a_n(w), \tag{1.38}$$

is the Laurent expansion about the point $z = w$, and $J^a_n(0) = J^a_n(w)$. Of course, we also have

$$J^a(w) = \frac{1}{2\pi i} \int_w dz (z-w)^n J^a(z), \tag{1.39}$$

where $\int_w dz$ will indicate integration around a contour enclosing the point $z = w$. This allows us to write

$$\begin{aligned}[J^a_m, J^b_n] &= \frac{1}{2\pi i}\int_c dww^n \frac{1}{2\pi i}\int_{|z|\geq|w|} dzz^m J^a(z)J^b(w) - \\ &\frac{1}{2\pi i}\int_c dww^m \frac{1}{2\pi i}\int_{|z|\leq|w|} dzz^m J^b(w)J^a(z). \end{aligned} \tag{1.40}$$

So, by subtraction of contours, and using (1.37), residue calculus then gives

$$[J^a_n, J^b_m] = \sum_c if^{abc}J^c_{n+m} + kn\delta^{ab}\delta_{m+n,0}. \tag{1.41}$$

Identical commutation relations hold for the current modes $\bar{J}^a_m$. These are the commutation relations of $\hat{g} \oplus \hat{g}$. It is easy to see that (1.41) is a central extension of the loop algebra of g. Consider $J^a \oplus s^n$, with s on the unit circle in the complex plane, and $n \in \mathbb{Z}$. The loop algebra of g is generated by the $J^a \oplus s^n$, since they are $g-$ valued functions on S^1. Now

$$[J^a \otimes s^m, J^b \otimes s^n] = [J^a, J^b] \otimes s^{n+m} = if^{abc}J^c \otimes s^{m+n} \tag{1.42}$$

so only the central extension term k is missing. The central extension term is known as a Schwinger term. (1.37) is not the usual form in quantum field theory, because radial quantisation is not typical. If we switch variables using $z = exp(\frac{2\pi i x}{L})$, then Laurent series become Fourier series, and we recover the more familiar form

$$[J'^a(x), J'^b(y)] = if^{abc}J'^c(x)\delta(x-y) + \frac{1}{2\pi}\delta^{ab}k\delta'(x-y), \tag{1.43}$$

where we have put

$$J'^a(x) := z\frac{J^a(z)}{L}.$$

The Schwinger term (and therefore, the presence of the level k [15]) is a quantum effect and is related to chiral anomalies. The conformal invariance of the model can now be established in a straightforward way. The Sugawara construction expresses the stress - energy tensor in terms of normal ordered products of currents $J^a(z)$. The Sugawara stress - energy tensor is

$$T(z) = \frac{1}{2(k+\bar{h}_g)} : J_a(z)J_a(z) : \tag{1.44}$$

where :: denotes normal ordering with respect to the modes of $J^a(z)$, and $\bar{h}_g$ is the dual Coxeter number of $\hat{g}$. Using the above information, we have the OPE

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.45}$$

with the central charge

$$c = \frac{kdimg}{k+\bar{h}_g}. \tag{1.46}$$

Each highest weight representations at level k is labelled by a vector p in the set

$$\{p \in P | \, p.\omega_i \geq 0, \, p.\rho \leq k\}, \tag{1.47}$$

where P is the weight lattice of $\hat{g}$, $\{\omega_i\}$ the corresponding fundamental weights, and ρ half the sum of the positive roots in $\hat{g}$. The primary field with highest weight p has conformal weight

$$\Delta p = \frac{p.(p+2\rho)}{k+\bar{h}_g} \tag{1.48}$$

Substituting $T(z) = \sum_{n\in\mathbb{Z}} z^{-2-n} L_n$ yields the usual form of the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{1.49}$$

For completeness, we also write

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} + \ldots \tag{1.50}$$

Expanding the currents in Laurent series, $J(z) = \sum J_n z^{-n-1}$ and likewise for $\bar{J}$, we can recover the affine Kac - Moody algebra,

$$[J^a_m, J^b_n] = f^{ab}_c J^c_{m+n} + \frac{1}{2} k m \delta_{m+n,0}\delta^{ab}, \tag{1.51}$$

which corresponds to

$$[L_m, J^a_n] = -nJ^a_{m+n}. \tag{1.52}$$

This shows that the $\hat{g}$ and Virasoro algebras extend to a semi - direct product in the theory. Furthermore, the full chiral algebra of the WZNW model consists of the Virasoro and Kac - Moody algebras. So the energy momentum tensor of the theory is bilinear in the current $J(z)$ and so its Laurent modes, the generators L_n of the Virasoro algebra, are bilinear in the Kac - Moody generators J^a_m. That tensor has the Sugawara form. This is a characteristic feature of WZNW models.

Witten and Gepner took [16] the WZNW model to describe strings propagating on group manifolds, so now we have a string theory with a large symmetry algebra, an affine Kac - Moody algebra and the Virasoro algebra. They showed that modular invariant partition functions of WZNW models can be constructed as bilinear in characters of the Kac - Moody algebras g, $\bar{g}$, which carry a finite representation of the modular group as shown by Kac and Peterson [17]. More particularly, for a given integer value of the level of the representation, there is a finite number of characters. A character under modular transformations will be a linear combination of all the characters at that level. It is generally argued that integer level is required for a single valued Wess -Zumino contribution to the exponential of the WZNW action.

However, one might investigate the consequences of allowing a fractional value for the level in the context of the WZNW model and of extended CFT. A geometric view of fractional level in WZNW models was provided in [18], and there are nowadays several contexts in which fractional level plays rôle. Here we mention three applications : $SL(2;\mathbb{R})$, $SL(2/1;\mathbb{R})$ and $N=2$.

First of all we start with the $SL(2;\mathbb{R})$ WZNW model. In the case where WZNW models are based on compact Lie groups such as SU(2) [14] [16], the unitary representations are finite - dimensional and correlation functions can be written as a finite sum of conformal blocks by the bootstrap approach. The non - compact groups give a much more complicated situation and the general solution is not known. There has been much work done on analysing this [19] [20]. We know that the group $SL(2;\mathbb{R})$ which is one of the simplest non- compact Lie groups, is important in many key areas. The first is in two - dimensional gravity [21] [22]. In that case the gravitational Ward identities for correlation functions of the metric are precisely the same as the $SL(2;\mathbb{R})$ Knizhnik - Zamolodchikov (KZ) equations [23].

The second application is the Quantum Hall Plateau Transition where the $SL(2;\mathbb{R})$ WZNW model has recently been proposed as a low energy effective field theory [24].

Finally, String theory on AdS_3 [25] [26] which is the $SL(2;\mathbb{R})$ group manifold, is described by a $SL(2;\mathbb{R})$ WZNW theory. The duality is between string theory in the bulk of AdS and conformal field theory (CFT) on the boundary of the space-time. Bulk fields naturally couple to local operators in the boundary CFT. Correlation functions on the boundary can be computed from the bulk theory in the super-gravity approximation by taking the classical tree level graphs for the bulk interactions. For general AdS_n, still there is an open problem because one does not know how to describe the full string theory in such a background. For the case of AdS_3 however the world sheet theory is described by the $SL(2;\mathbb{R})$ WZNW model and the boundary theory is a two dimensional CFT . In that case fields are naturally classified according to the representation theory of $SL(2;\mathbb{R})$. We have unitary representations and the string theory is exactly solvable in principle.

In my thesis we investigate further the characters of the complex affine superalgebra $\hat{sl}(2|1;\mathbb{C})$ and their modular transformations, because our results will be useful in determining the exact nature of the correspondence between the theory of N=2 non - critical strings and the $SL(2|1;\mathbb{R})/SL(2|1;\mathbb{R})$gauged WZNW model [27]. Such theories are certainly not studied in this work. The essential ingredients of the WZNW theory are encoded in its current algebra, the Kac - Moody algebra. The exact correspondence between the traditional approach to noncritical string and G/G models is yet to be proven. However, a crucial ingredient in the description of the spectrum in the G/G picture is the representation theory of the corresponding affine Lie algebra, $\hat{g}$, at fractional level $k = \frac{p}{u} - \bar{h}_g$, $p \in \mathbb{Z}^*$, $u \in \mathbb{N}$ and $gcd(p, u) = 1$ with $\bar{h}_g$ the dual Coxeter number of $\hat{g}$. For instance, the $SL(2|1;\mathbb{R})/SL(2|1;\mathbb{R})$ topological quantum field theory obtained by gauging the anomaly free diagonal subgroup $SL(2|1;\mathbb{R})$ of the global $SL(2|1;\mathbb{R})_L \otimes SL(2|1;\mathbb{R})_R$ symmetry of the WZNW model appears to be intimately related to the noncritical charged fermionic string, which is the prototype of $N = 2$ super-gravity in two dimensions. A comparison of the ghost content of the two theories strongly suggests that the $N = 2$ noncritical string is equivalent to the tensor product of twisted $SL(2|1;\mathbb{R})/SL(2|1;\mathbb{R})$ WZNW model.

It is however only when a one- to- one correspondence between the physical states and the equivalence of the correlation functions of the two theories are established that one can view the twisted G/G model as the topological version of the corresponding noncritical string theory. Our original motivation in the analysis of the representation theory of $\hat{sl}(2|1;\mathbb{C})$ at fractional level k is its potential relevance in the description of $N=2$ super-strings. We take the matter coupled to super-gravity in an $N=2$ super Coulomb gas representation with central charge

$$c_{matter} = 3(1-\frac{2p}{u}),\ p,u\in\mathbb{N},\ gcd(p,u)=1. \tag{1.53}$$

The level k of the affine super-algebra $\hat{sl}(2|1)$ appearing in the $SL(2|1;\mathbb{R})/SL(2|1;\mathbb{R})$ gauged WZNW model, believed to be intimately related to the $N=2$ string, is of the form

$$k=\frac{p}{u}-1. \tag{1.54}$$

This is precisely the type of fractional level first discussed in the paper by Kac and Wakimoto on admissible representations of affine Lie algebras [28]. For levels of the form (1.54) with $p=1$, one obtained a description of unitary minimal $N=2$ matter whose spectrum is described by a finite number of irreducible representations, whose characters form a finite representation of the modular group [29]. There are thus rational, although non unitary, theories associated with $\hat{sl}(2|1;\mathbb{C})$ at fractional level, as we shall discuss in chapters 3 and 4.

1.4 Modular Transformations

Modular invariance has recently emerged as a powerful tool in the study of conformal field theory in two dimensions. Constraints from modular invariance make it possible to determine the exact spectrum of the theories, with various boundary conditions. Such constraints were found in the work of Cardy [30] for the minimal conformal models [8]. Modular invariance for the WZNW theories was analysed by Gepner and Witten [16], as part of study of string propagation on group manifolds. The super-conformal models [31] were treated by Kastor [32].

In this section we first express the modular transformation. For applications in string theory one needs to consider conformal field theories that are defined on general Riemann surfaces rather then on the complex plane. Conformal field theory is described algebraically by a set of states. One would expect all the states in a theory to contribute to loop diagrams. For this reason we are going to study conformal field theory on the simplest loop diagram, the torus. In (1.17) h is the genus of the surface and g_s is the string coupling constant. String perturbation theory is a summation over two - dimensional surfaces. This sum splits in a sum over all different topologies, and integrals over all different shapes of surfaces with a given topology, the moduli. This is analogous to Feynman diagrams with different numbers of loops in ordinary field theory. In two dimensions the topology can be described by a single parameter (the number of handles or genus). At genus 1 (torus) there is one complex modulus, the parameter τ. The integral over τ is not over the full positive upper half plane, but should be restricted to a region that covers the set of distinct tori just once. The entire upper half plane is covered with an infinite number of regions with different shapes and sizes. The integral over τ should not depend on the choice of the region, or otherwise the theory is not well - defined. If the theory is modular invariant, this problem does not arise. So modular transformations change the value of the moduli but not the shape of the surface. Now we describe the torus in terms of a lattice defined by two basis vectors, corresponding to the points 1 and τ in the complex plane. However, the same lattice and the same torus can be described just as well by choosing different basis vectors. One should keep in mind that the torus was defined by aligning one basis vector along the real axis in the complex plane, and scaling it to one, but one could have chosen instead the direction τ. This has the effect of replacing τ by $\frac{-1}{\tau}$. This is most easily illustrated by taking τ purely imaginary. The set of such transformations of the torus forms a group, called the modular group.

We have identified two elements of that group, namely

$$\begin{aligned} T &: \tau \to \tau + 1, \\ S &: \tau \to \tfrac{-1}{\tau}, \end{aligned} \tag{1.55}$$

It turns out that these two transformations generate the entire group.

The most general modular transformation has the form

$$\begin{aligned} \tau \to \tfrac{a\tau+b}{c\tau+d},\ a,b,c,d \in \mathbb{Z}; \\ ad-bc=1. \end{aligned} \tag{1.56}$$

It is now natural to ask how the partition function behaves under transformations in the modular group. If we start with a well - defined two - dimensional theory on the torus, in which all fields are periodic along all cycles around the torus, the result of the path - integral should not depend on how that torus was parametrised. Hence the partition function should be invariant under modular transformations. If we compute the path integral for (1.17) on the torus, we will automatically get a modular invariant partition function. On the other hand, if we verify a partition function written in terms of characters is modular invariant we need to know how the characters transform.

Kac and Peterson [17] have shown how nicely the characters transform under modular transformations.

1.5 A pedestrian approach to Admissible Representations of Affine Lie Algebras

We try to give first an informal and hopefully intuitive version of admissibility in the case of the well - known affine Lie algebra $\widehat{su}(2)$ at level k, and introduce the conventional formalism of representation theory in order to provide the reader with the tools necessary for reading the mathematical literature on this subject [35]. The book by Di Francesco, Mathieu and Sénéchal [33] is a very good source of inspiration, as well as a few papers by Mathieu and Walton [34].

Consider $\widehat{su}(2)$ at generic level k, whose commutation relations are given by

$$[J^i_m, J^j_n] = \epsilon^{ijk} J^k_{m+n} + mk\delta_{m+n,0}\delta^{ij}, \tag{1.57}$$

for $m,n \in \mathbb{Z}$ and $i,j,k = 1,2,3$. We will use the complexified version of the algebra, with the step operators $J^{\pm}_m = J^1_m \pm iJ^2_m$, and the Cartan generator J^3_0. The commutation relations become

$$\begin{aligned} [J^+_m, J^-_n] &= 2J^3_{m+n} + km\delta_{m+n,0}, \\ [J^3_m, J^\pm_n] &= \pm J^\pm_{m+n}, \\ [J^3_m, J^3_n] &= mk\delta_{m+n,0}. \end{aligned} \tag{1.58}$$

A highest weight state $|\Omega>$ of $\widehat{su}(2)_k$ satisfies

$$\begin{aligned} J^+_n|\Omega> &= 0, \qquad \forall n \geqslant 0, \\ J^-_n|\Omega> &= 0, \qquad \forall n \geqslant 1, \\ J^3_n|\Omega> &= 0, \qquad \forall n \geqslant 1, \end{aligned} \tag{1.59}$$

and we will label j the J^3_0 - eigenvalue of $|\Omega>$: $J^3_0 = j|\Omega>$. Using (1.58)(1.59), one can show

$$\begin{aligned} J^+_0(J^-_0)^n|\Omega> = n(2j+1-n)(J_0)^{n-1}|\Omega>, \\ n \in \mathbb{N} \ (ie \ n = 1, 2, 3, ...). \end{aligned} \tag{1.60}$$

The smallest nonzero integer n for which the above expression vanishes is

$$n = 2j + 1, \tag{1.61}$$

which has a solution if $j \in \frac{1}{2}\mathbb{Z}_+ \qquad (\mathbb{Z}_+ = \{0, 1, 2, ...\})$. Consider

$$J^-_{+1}(J^+_{-1})^{n'}|\Omega> = (k+1-2j-n')(J^+_{-1})^{n'-1}|\Omega>, \quad n' \in \mathbb{N}. \tag{1.62}$$

The smallest nonzero integer n' for which the above expression vanishes is

$$n' = k + 1 - 2j. \tag{1.63}$$

Given (1.61), the equality (1.63) is possible if k is an integer satisfying

$$k \geqslant 0.$$

When (1.61) and (1.63) hold, there exist two primitive singular vectors $(J^-_0)^{2j+1}|\Omega>$ and $(J^+_{-1})^{(k-2j+1)}|\Omega>$ in the Verma module built from $|\Omega>$. A direct consequence is that, for k a positive integer, there exist $(k+1)$ irreducible highest weight representations labelled by $j = 0, \frac{1}{2}, ..., \frac{k}{2}$. In each of them, the weights organise themselves into finite representations of the "horizontal" algebra $su(2)$. The crucial powers $2j+1$ and $k+1-2j$ can be re-expressed in a formalism which allows generalisation to other Lie groups.

It is conventional to describe the simple roots of $\widehat{su}(2)$ as

$$\begin{aligned}\alpha_0 &= (-\alpha; 0, 1),\\ \alpha_1 &= (\alpha; 0, 0).\end{aligned} \tag{1.64}$$

In general, the components of the affine weight λ at level k are given by

$$\lambda = (\bar{\lambda}; k_\lambda, n_\lambda), \tag{1.65}$$

where $\bar{\lambda}$ is the finite part of λ (it is a weight in the Lie algebra $su(2)$), k_λ is the level and n_λ refers to $(-L_0)$. The scalar product (λ, ν) is given by

$$(\lambda, \nu) = (\bar{\lambda}, \bar{\nu}) + k_\lambda n_\nu + k_\nu n_\lambda. \tag{1.66}$$

In (1.64), the $su(2)$ simple root α is normalised as $\alpha^2 = 2$. One associates the generator J^-_{+1} to α_0, and J^+_0 to α_1. The coroots are given by $\alpha_0^\vee = \frac{2}{\alpha_0^2}\alpha_0$ and $\alpha_1^\vee = \frac{2}{\alpha_1^2}\alpha_1$, and consequently, the fundamental weights are

$$\Lambda_0 = (0; 1, 0), \qquad \Lambda_1 = (\tfrac{1}{2}\alpha; 1, 0). \tag{1.67}$$

It is customary to parametrise an arbitrary weight Λ at level k as

$$\Lambda = (k - 2j)\Lambda_0 + 2j\Lambda_1, \tag{1.68}$$

where $n_0 = k - 2j$ and $n_1 = 2j$ are the Dynkin labels associated to Λ. When those are non-negative integers, the representation with highest weight Λ is integrable, which means in particular, that Λ generates an "horizontal" representation which is finite - dimensional.

The co-marks $a_0^\vee$ and $a_1^\vee$ satisfy $k = a_0^\vee n_0 + a_1^\vee n_1$ (in our case of $\widehat{su}(2)_k$, $a_0^\vee = a_1^\vee = 1$) and their sum is the dual Coxeter number

$$\bar{h}_g = a_0^\vee + a_1^\vee = 2.$$

They also provide an expression for the canonical central element

$$K = \sum_{j=0}^{1} a_j^\vee \alpha_j^\vee. \tag{1.69}$$

In order to rewrite $(k - 2j + 1)$ and $2j + 1$ in the language of roots and weights, we also need to introduce the Weyl vector

$$\rho = \Lambda_0 + \Lambda_1. \tag{1.70}$$

It is now easy to see that

$$
\begin{aligned}
(\alpha_0^\vee, \Lambda+\rho) &= (\alpha_0^\vee, (k-2j)\Lambda_0 + 2j\Lambda_1 + \Lambda_0 + \Lambda_1) \\
(\alpha_1^\vee, \Lambda+\rho) &= (\alpha_1^\vee, (k-2j)\Lambda_0 + 2j\Lambda_1 + \Lambda_0 + \Lambda_1).
\end{aligned}
\tag{1.71}
$$

So the condition $(\alpha,\ \Lambda+\rho) \in \mathbb{N}\ \forall\alpha \in \prod^\vee$ ($\prod^\vee$ is the set of coroots, ie $\prod^\vee = \{\alpha_0^\vee,\ \alpha_1^\vee\}$) is crucial in ensuring the integrability of representations. Note that it is the highest weight shifted by the Weyl vector ρ which enters the condition. Since $(k, \Lambda+\rho) = k + \bar{h}_g$ it will come as little surprise that the shifted level $(k+\bar{h}_g)$ also plays a rôle in the admissible case.

In general, denoting by $X_{r,k}$ the untwisted affine algebra $X_r^{(1)}$ of rank $(r+1)$ and level k, the set of integrable highest weights is

$$
P_+^k = \{\lambda | (\alpha, \lambda) \in \mathbb{Z}_+ \ \ \forall\alpha \in \overset{\vee}{\prod};\ \ (K,\lambda) = k\}
\tag{1.72}
$$

Next we discuss characters. One defines the character of an integrable representation $M(\lambda)$ of highest weight λ by

$$
Ch_\lambda = \sum_{\sigma\in P} mult_\lambda(\sigma) e^\sigma,
\tag{1.73}
$$

where P is the weight lattice.

The elegant Weyl - Kac formula for characters is,

$$
Ch_\lambda = \frac{\sum_{w\in W} det(w) e^{w.\lambda}}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{mult(\alpha)}}.
\tag{1.74}
$$

Δ_+ is the set of positive roots and $w.\lambda = w(\lambda+\rho) - \rho$ is the shifted action of w, and W is the affine Weyl group.

The normalised character is

$$
X_\lambda = e^{-\delta(h_\lambda - \frac{c}{24})} Ch_\lambda,
\tag{1.75}
$$

where the conformal weight of the primary field λ is

$$
h_\lambda = \frac{(\lambda, \lambda+2\rho)}{2(k+\bar{h}_g)} = \frac{|\lambda+\rho|^2 - |\rho|^2}{2(k+\bar{h}_g)},
\tag{1.76}
$$

with central charge

$$
C_{X_r} = \frac{k dim X_r}{(k+\bar{h}_g)}.
\tag{1.77}
$$

$\bar{h}_g$ stands for the dual Coxeter number, k for the level of the affine algebra.

Suppose that a weight σ has imaginary part - $n\delta$. Define

$$e^{\sigma}(\tau, z, t) := exp[2\pi i(n\tau + (\sigma|z) + kt)],$$

where z in an element of the Cartan subalgebra of the horizontal subalgebra X_r. Then the normalised characters $X_\lambda(\tau, z, t)$ are the conformal blocks for the torus partition function of the WZNW conformal field theory. Kac and Peterson [17] have shown how the characters transform under modular transformations. We can see

$$X_\lambda(-\frac{1}{\tau}, \frac{z}{\tau}, \frac{y-(z|z)}{2\tau}) = \sum_{\mu \in P_+^k} S_{\lambda,\mu}^k X_\mu(\tau, z, y), \tag{1.78}$$

and

$$X_\lambda(\tau + 1, z, y) = \sum_{\mu \in P_+^k} T_{\lambda,\mu}^k X_\mu(\tau, z, y), \tag{1.79}$$

The $S_{\lambda,\mu}^k (T_{\lambda,\mu}^k)$ are the elements of a unitary, symmetric matrix $S^k(T^k)$.

What if one relaxes the condition that k is a positive integer? Let us come back to $\widehat{su(2)}_k$. If k becomes fractional, some Dynkin labels will become fractional, leading in some cases to infinite - dimensional representations of the " horizontal " algebra $\widehat{su}(2)$ (when $2j \notin \mathbb{Z}$).

As Kac and Wakimoto discovered however [28] [35] [36] , the situation is " almost " as pleasant when the level k is given by

$$k = \frac{t}{u}, \ gcd(t, u) = 1, \ t \in \mathbb{Z}^*, \ u \in \mathbb{N} \tag{1.80}$$

as in the integrable ($u = 1$) case. Since we already noticed that $(k + \bar{h}_g)$ plays an important rôle in the integrable case, and since we expect $2j+1$ to become fractional for some values of j at fixed level k, let us write

$$2j + 1 = n - n'(k + \bar{h}_g) \tag{1.81}$$

with $0 \leq n' \leq u-1$ and $n \in \mathbb{Z}$. This parametrisation isolates an integer contribution to $2j+1$ (note that $n'\bar{h}_g \in \mathbb{Z}_+$ and $n'[k] \in \mathbb{Z}$, for $[k]$ the integer part of k, so that the integer contribution n is not the integer part of $2j+1$).

Why is this parametrisation useful? With (1.81), we can write (1.68) as

$$\begin{aligned}\Lambda &= [k+1-n+n'(k+\bar{h}_g]\Lambda_0 + [n-1-n'(k+\bar{h}_g)]\Lambda_1 \\ &= \{[1-n-\bar{h}_g+u(k+\bar{h}_g)]\Lambda_0+[n-1]\Lambda_1\} - (k+\bar{h}_g)\{[u-1-n']\Lambda_0+n'\Lambda_1\} \\ &\equiv \Lambda^I-(k+\bar{h}_g)\Lambda^F, \end{aligned} \tag{1.82}$$

where $\Lambda^F = (u-1-n')\Lambda_0 + n'\Lambda_1$ is an integrable weight at level $u-1 \geq 0$ (in view of the n' range in (1.81) and the u in (1.80)). On the other hand, the weight $\Lambda^I = (1-n-\bar{h}_g+u(k+\bar{h}_g))\Lambda_0 + (n-1)\Lambda_1$ is integrable provided

$$1 \leq n \leq 1-\bar{h}_g+u(k+\bar{h}_g), \tag{1.83}$$

which implies $\quad u(k+\bar{h}_g)-\bar{h}_g \geq 0.$

Bearing in mind that $\bar{h}_g = 2$, we conclude that, for j parametrised as

$$2j_{nn'}+1 = n-n'(k+2)\ , \qquad \begin{aligned} 1 \leq n &\leq t+2u-1 \\ 0 \leq n' &\leq u-1, \end{aligned} \tag{1.84}$$

one may split a non - integrable weight at level $k = \frac{t}{u}$ in two integrable weights at levels $u(k+2)-2$ and $u-1$ according to the formula

$$\Lambda = \Lambda^I-(k+\bar{h}_g)\Lambda^F. \tag{1.85}$$

All admissible weights in $\widehat{su}(2)_k$ can be put in the form (1.85). When $u=1$, $n'=0$ and $2j+1=n$, $1 \leq n \leq t+1$ and one recovers the integrable case, with $\Lambda^F = 0$, $k^I = k = t$.

For higher rank groups, an admissible level k weight Λ may be rewritten as

$$\Lambda = y.(\Lambda^I-(k+\bar{h}_g)\Lambda^{F,y}), \tag{1.86}$$

where Λ^I and $\Lambda^{F,y}$ are both integrable at levels $k^I = u(k+\bar{h}_g)-\bar{h}_g \geq 0$ and $k^F = u-1 \geq 0$ respectively.

The new ingredient when comparing with $\widehat{SU}(2)_k$ is the action of a non - trivial element y of the subgroup $W/W(A)$ where W is the finite Weyl group and $W(A)$ is the subgroup of W isomorphic to the outer - automorphism group of $X_{r,k}$. We refer to Di Francesco [33] for a more complete analysis of this point. Because Λ is built from two integrable weights at finite possible levels, there exists a finite number $(k^I+1).(k^F+1)$ of admissible representations at level $k = k^I - (k+\bar{h}_g)k^F$.

In this thesis, we have used the following parametrisation of j for $\widehat{su}(2)$, $k+2=\frac{p}{u}$

$$j(r,s) = \tfrac{1}{2}(r-1) - \tfrac{1}{2}(s-1)\tfrac{p}{u}, \qquad \begin{cases} 1 \le r \le p-1, \\ 1 \le s \le u. \end{cases} \tag{1.87}$$

The conformal dimension of the highest weight with isospin $j(r,s)$ is

$$h(r,s) = \frac{j(r,s)[j(r,s)+1]}{k+2}, \tag{1.88}$$

and is negative for some values of r and s, which is a very strong sign of non - unitarity.

Although non - unitary, the admissible representations share many important properties with integrable representations. For instance, their characters obey a generalisation of the Weyl - Kac formula and they have " nice " modular properties. By this we mean that in the case of $\widehat{su}(2)_k$, $k=\frac{t}{u}$, the admissible characters (and there is a finite number of them) transform covariantly under the action of the modular group.

This property percolates to the coset theories

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}, \tag{1.89}$$

which are non - unitary Virasoro theories with

$$\begin{aligned} c &= \tfrac{3k}{k+2} + 1 - \tfrac{3(k+1)}{k+3} = 1 - \tfrac{6}{(k+2)(k+3)} \\ &= 1 - \tfrac{6u^2}{p(p+u)} = 1 - \tfrac{6(p-p')^2}{pp'}, \end{aligned} \tag{1.90}$$

where we used $k+2=\frac{p}{u}$ and $p'=p+u$.

The main object of this thesis has been to derive the modular transformations of the admissible characters of the $\widehat{sl}(2|1)$ super-algebra at fractional level $k+1=\frac{p}{u}$, $p \in \mathbb{Z}^*$, $u \in \mathbb{N}$, $(p,u)=1$ $(p,u$ coprime).

To this effect, we have heavily exploited the $\widehat{sl}(2)_k$ content of $\widehat{sl}(2|1)_k$, as well as sumrules which allow to re-express a sum of products of $\widehat{sl}(2|1)_k$ characters with $\widehat{U}(1)$ characters as a sum of triple products of $\widehat{sl}(2)$ at levels

$$\begin{aligned} k &: k+1 = \tfrac{p}{u} \\ k' &: k'+1 = \tfrac{u}{p}, \end{aligned} \tag{1.91}$$

and 1.

Here the level k is shifted by $\bar{h}_g$ and the level k' becomes integer when $p = 1$. In that case, we have shown the covariance of admissible $\widehat{sl}(2|1)_k$ characters under the action of the modular group using the sumrules. We can confirm the embryonic work of M. Hayes [37] and the follow up in G. Johnstone's [38], where the modular transformations were obtained by a different method.

The $\widehat{sl}(2|1)$ algebra is intimately related to the superconformal algebra (SCA) $N = 2$. The latter may be obtained from the former by Hamiltonian reduction [39]. At the level of characters, one notes that a subset of admissible $\widehat{sl}(2|1)_k$ characters develop simple poles when the angular variable $z = e^{2\pi i\mu} \to 1$. The residues at these poles are proportional to the $N = 2$ SCA characters, at $c = 3(1 - 2\frac{p}{u})$. When $p = 1$, one recognises the unitary minimal $N = 2$ series, and the nice modular properties of $\widehat{sl}(2|1)_k$ characters are inherited by the corresponding unitary $N = 2$ characters. However, when $p \neq 1$, the minimal $N = 2$ characters obtained by residuing are non - unitary and do not transform as nicely under the modular group as their $N = 0$ counterpart, as we establish in Chapter 2.

There, the modular properties were derived by exploiting branching rules of $\widehat{sl}(2)_k$ into $N = 2$ characters, but the results can also be obtained indirectly, by taking the appropriate residues in the modular transformed $\widehat{sl}(2|1)_k$ characters which can be found in Chapter 4.

When $p \neq 1$, the $\widehat{sl}(2|1)$ admissible characters are no more periodic in the spectral flow parameter and consequently, one has to face the complication of an infinite family of admissible characters, which transform covariantly up to extra terms which vanish when $p = 1$, but whose presence should receive an elegant interpretation fairly soon .

In conclusion, this thesis contributes to the study of conformal field theory in the non - unitary sector. By using a pedestrian method which required however some specific skills to implement, we were able to challenge our collaborators who are deriving the same results with much more powerful techniques, providing them with reliable expressions against which results can be compared and merged [40].

1.6 Layout of the thesis

The remaining chapters of this thesis split naturally into two parts. The first part consists of just Chapter 2, and Chapters 3 and 4 form the second part.

In Chapter 2 we give a detailed review of the calculation of modular transformations of $N = 2$ characters. The chapter treats both the unitary and non - unitary minimal cases. In the former case, the character formula has a nice periodicity under spectral flow, so we can easily write modular transformations for N = 2 characters. This result agrees with Ravanini and Wakimoto [41] [42]. I have also computed modular transformations for N = 2 characters in non - unitary case. In that case we had several difficulties, because the $N = 2$ characters under spectral flow are linear combinations of an infinite number of characters. We could hardly solve this problem. The results of this chapter are new and have not appeared before in the literature. The results of this chapter were impotant for us to solve the modular transformations for $\widehat{sl}(2|1)$ in Chapter 4.

Characters for $\widehat{sl}(2|1)$ at fractional level are presented in Chapter 3. The *complete set* was first obtained for general p by A. Semikhatov and A. Taormina.

Chapter 4 deals with nice sumrules one can write for $\widehat{sl}(2|1)$ characters. These have enabled me to calculate their S modular transformation at general level $k = \frac{t}{u}$ (the T transform is straightforward).

All results in chapters 2 and 4 are new and will be published soon [40].

Chapter 2

The $N = 2$ Superconformal Algebra

2.1 Introduction

Supersymmetric extensions of two-dimensional conformal symmetry play an important rôle in the formulation of superstring theories and in various statistical mechanics models. The first example of superconformal symmetry was developed by Neveu and Schwarz [43], and also Ramond [44] when they constructed fermionic string models. This 'world-sheet' supersymmetry corresponds to the $N = 1$ superconformal algebra (SCA).

A natural generalisation of the above SCA consists in increasing the number N of supersymmetry generators. The closure of the N-extended conformal algebra ($N \geq 2$) usually requires additional bosonic generators corresponding to the Kac-Moody symmetries that rotate the supersymmetry generators among themselves. The $N = 2$ SCA is the simplest non-minimally extended conformal algebra, having an $O(2)$ Kac-Moody subalgebra. Originally, it appeared in the formulation of the 'spinning' string, a fermionic string with extended world-sheet supersymmetry [45, 46]. However, it subsequently found beautiful applications in the study of space-time supersymmetric compactifications of ten-dimensional superstrings to four dimensions [47, 48]. It is also believed to play a rôle in some two-dimensional statistical systems at criticality [49].

In most string applications so far, the unitary representations of the $N = 2$ SCA have been used. This explains the huge literature available on this particular class of representations, the corresponding characters and the construction of modular invariant partition functions [50] [51] [52]. Unitary representations occur for discrete values of the central charge in the series,

$$c(u,1) = 3(1 - \frac{2}{u}), \qquad u = 3, 4, \cdots, \tag{2.1}$$

or for continuous values of c in the range $c \geq 3$. However, in the context of non-critical superstrings [53] [54] [55] as well as in two-dimensional critical phenomena, non-unitary representations of $N = 2$ come into play. From a more mathematical point of view, the representation theory in the non-unitary discrete sector has not been much developed so far. Irreducible representations occur for central charges in the series,

$$c(u,p) = 3(1 - \frac{2p}{u}), \qquad u = 3, 4, \cdots \quad p = 2, 3, \cdots, u \text{ and } p \text{ coprime}, \tag{2.2}$$

but the corresponding characters do not close under the action of the modular group, in striking contrast with the unitary case.

The aim of this chapter is to study the behaviour of these $N = 2$ non-unitary characters under the action of the modular group and to highlight the differences with the unitary characters. In section 2, we review the basic structure of the $N = 2$ SCA. In section 3, we introduce a special class of irreducible highest weight representations of the $N = 2$ SCA which are non-unitary and called *admissible*, and present their character formulas. Section 4 reviews some fundamental features of spectral flow and studies periodicity properties of characters. Finally, in section 5, we study the behaviour of characters under the modular group.

2.2 The $N = 2$ Superconformal Algebra

The full $N = 2$ SCA is the direct sum of two copies of the algebra we write down below, and corresponds to the analytic and anti-analytic components of the $N = 2$ currents. Throughout this thesis, we concentrate on the analytic sector when discussing representation theory, and losely call '$N = 2$ SCA' the analytic sector of the full algebra.

The $N = 2$ SCA is generated by two bosonic currents (the stress energy tensor

$T(z)$ of conformal dimension two and the $U(1)$ current $J(z)$ of conformal dimension one), and two supercharges $G^{\pm}(z)$ of conformal dimension 3/2. The complex plane variable z is related to the string coordinates on the cylindrical worldsheet by $z = e^{\tau+i\sigma}$, with $\tau \in \mathbb{R}$ and $0 \leq \sigma \leq 2\pi$.

The (non regular) operator product expansions defining the $N = 2$ SCA are [56],

$$\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots, \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \cdots, \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} + \cdots, \\
T(z)G^{\pm}(w) &= \frac{3G^{\pm}(w)}{2(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w} + \cdots, \\
J(z)G^{\pm}(w) &= \pm\frac{G^{\pm}(w)}{z-w} + \cdots, \\
G^{+}(z)G^{-}(w) &= \frac{2c}{3(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \frac{2T(w)}{z-w} + \cdots.
\end{aligned} \tag{2.3}$$

The above current algebra is invariant under $O(2)$ rotations of the two supercharges $G^1 = G^+ + G^-$ and $G^2 = i(G^+ - G^-)$, namely under the transformations,

$$\begin{pmatrix} G^1 \\ G^2 \end{pmatrix} \to \mathcal{O} \begin{pmatrix} G^1 \\ G^2 \end{pmatrix} \tag{2.4}$$

where $\mathcal{O}$ is a 2×2 orthogonal matrix. The $SO(2) \equiv U(1)$ continuous subsymmetry allows for the boundary conditions,

$$G^{\pm}(e^{2i\pi}z) = e^{\mp 2i\pi\theta}G^{\pm}(z), \tag{2.5}$$

$$T(e^{2i\pi}z) = T(z), \qquad J(e^{2i\pi}z) = J(z), \tag{2.6}$$

with θ a continuous parameter in the range $0 \leq \theta \leq 1$. The above boundary conditions determine the mode expansions of all currents. Let us discuss the supercharges in some detail. The Laurent expansion of $G^{\pm}(z)$ is given by,

$$G^{\pm}(z) = \sum_{k \in \mathbb{Z} \pm \theta - 3/2} G^{\pm}_k z^{-k-3/2} = \sum_{n \in \mathbb{Z}} G^{\pm}_{n \pm \theta + 1/2} z^{-n \mp \theta - 2}. \tag{2.7}$$

If $\theta = 0$, $G^{\pm}$ have half-integer modes and the fields are single-valued. This corresponds to the Neveu-Schwarz (NS) sector of the theory, as the fields defined *on the* *cylinder* are anti-periodic for $\sigma \to \sigma + 2\pi$. Indeed, one has,

$$G^{\pm}(\tau + i\sigma + 2i\pi) = \sum_{n\in\mathbb{Z}} G^{\pm}_{n+1/2} e^{-(n+2)(\tau+i\sigma+2i\pi)} = G^{\pm}(\tau + i\sigma). \tag{2.8}$$

On the other hand, when $\theta = 1/2$, $G^{\pm}$ have integer modes and the fields have a branch cut. This corresponds to the Ramond sector of the theory as the fields defined on the cylinder are periodic. All continuous intermediate values of θ provide isomorphic $N = 2$ algebras. This is clearly seen when expressing the OPE of the currents (2.3) in terms of (anti)-commutation relations between their modes. To fix ideas, let us work in the Ramond sector, i.e. let us take $\theta = 1/2$. Note that (2.6) requires,

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \\ J(z) &= \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \end{aligned} \tag{2.9}$$

and therefore, the Ramond $N = 2$ SCA reads,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, J_n] &= -nJ_{m+n}, \\ [J_m, J_n] &= \frac{c}{3} m\delta_{m+n,0}, \\ \left[L_m, G^{\pm}_r\right] &= (\frac{m}{2} - r)G^{\pm}_{m+r}, \\ \left[J_m, G^{\pm}_r\right] &= \pm G^{\pm}_{m+r}, \\ \left[G^{+}_r, G^{-}_s\right]_+ &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}. \end{aligned} \tag{2.10}$$

The transformation

$$\begin{aligned} U_\theta : L_n &\mapsto L_n + \theta J_n + \frac{c}{6}\theta^2\delta_{n,0}, & J_n &\mapsto J_n + \frac{c}{3}\theta\delta_{n,0}, \\ G^{+}_n &\mapsto G^{+}_{n+\theta}, & G^{-}_n &\mapsto G^{-}_{n-\theta}, \end{aligned} \tag{2.11}$$

is an isomorphism of the above algebra and is called *spectral flow*. It continuously connects the Ramond algebra ($\theta = 1/2$) to the Neveu-Schwarz algebra ($\theta = 0$).

We will only be concerned here with the infinite family of isomorphic algebras parametrised by θ, and in particular with the Ramond (R) and Neveu-Schwarz (NS) algebras. Let us however mention that there exists a *twisted* $N = 2$ SCA which is obtained by exploiting the Z_2 symmetry which interchanges G^+ and G^-. The latter is nothing else than the transformation (2.4) when $\mathcal{O}$ is the orthogonal matrix

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.12}$$

and therefore allows for opposite boundary conditions for G^1 and G^2.

2.3 Admissible $N = 2$ representations and their characters

In order to make contact with the works of Semikhatov and collaborators, we work in the new basis,

$$\mathcal{L}_n = L_n + \frac{1}{2}(n+1)J_n, \qquad \mathcal{G}_n^{\pm} = G^{\pm}_{n\pm\frac{1}{2}}, \qquad \mathcal{J}_n = J_n, \tag{2.13}$$

which yields the following $N = 2$ commutation relations,

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, \\
[\mathcal{L}_m, \mathcal{J}_n] &= -n\mathcal{J}_{m+n} + \frac{c}{6}(m^3+m)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{J}_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
\left[\mathcal{L}_m, \mathcal{G}_r^+\right] &= (m-r)\mathcal{G}^+_{m+r}, \\
\left[\mathcal{L}_m, \mathcal{G}_r^-\right] &= -r\mathcal{G}^-_{m+r}, \\
\left[\mathcal{J}_m, \mathcal{G}_r^{\pm}\right] &= \pm\mathcal{G}^{\pm}_{m+r}, \\
\left[\mathcal{G}_r^+, \mathcal{G}_s^-\right]_+ &= 2\mathcal{L}_{r+s} + (r-s)\mathcal{J}_{r+s} + \frac{c}{3}(r^2+r)\delta_{r+s,0},
\end{aligned} \tag{2.14}$$

with $m, r \in \mathbb{Z}$. In this new basis, the spectral flow acts as follows ($\theta \in \mathbb{Z}$),

$$\begin{aligned}
U_\theta : \mathcal{L}_n &\mapsto \mathcal{L}_n + \theta\mathcal{J}_n + \frac{c}{6}(\theta^2+\theta)\delta_{n,0}, & \mathcal{J}_n &\mapsto \mathcal{J}_n + \frac{c}{3}\theta\delta_{n,0}, \\
\mathcal{G}_n^+ &\mapsto \mathcal{G}^+_{n+\theta}, & \mathcal{G}_n^- &\mapsto \mathcal{G}^-_{n-\theta}.
\end{aligned} \tag{2.15}$$

We will call the spectral flow parameter θ a *twist*.

We focus in this thesis on a particular class of $N = 2$ irreducible highest weight representations called *admissible*. They exist when the central charge is given by

(2.2). In order to characterise them, we first define a twisted highest weight vector $|h, c; \theta>$ as a vector satisfying the annihilation conditions,

$$\mathcal{G}^{+}_{\theta+m}|h, c; \theta> = \mathcal{G}^{-}_{-\theta+m}|h, c; \theta> = \mathcal{L}_{m+1}|h, c; \theta> = \mathcal{J}_{m+1}|h, c; \theta> = 0, \qquad (2.16)$$

with

$$\begin{aligned} (\mathcal{J}_0 + \frac{c}{3}\theta)|h, c; \theta> &= h|h, c; \theta> \\ (\mathcal{L}_0 + \theta\mathcal{J}_0 + \frac{c}{6}(\theta^2 + \theta))|h, c; \theta> &= 0. \end{aligned} \qquad (2.17)$$

A twisted Verma module is a module freely generated from a twisted highest weight vector $|h, c; \theta>$ by $\mathcal{G}^{+}_{-m+\theta}, \mathcal{G}^{-}_{-m-\theta}, \mathcal{J}_{-m}$ and $\mathcal{L}_{-m}$ with m a strictly positive integer. The admissible $N = 2$ highest weight representations are the quotients of twisted highest weight Verma modules over maximal submodules whose highest weights are singular vectors.

The corresponding characters were obtained in [57] and were also privately disclosed to us by [58]. They are labelled by integers r, s, and θ such that

$$1 \leq r \leq u - 1, \qquad 1 - p \leq s \leq p, \qquad \theta \in \mathbb{Z}. \qquad (2.18)$$

The quantum number h appearing in (2.17) is related to r, s and θ by,

$$h = s - 1 - (r - 1)\frac{p}{u} + \theta(1 - \frac{2p}{u}). \qquad (2.19)$$

The untwisted Ramond characters are formally defined as a trace over untwisted highest weight irreducible modules as,

$$X^{N=2}_{r,s,u,p}(z, q) = Tr(q^{\mathcal{L}_0} z^{\mathcal{J}_0}), \qquad (2.20)$$

where q and z are two complex variables, $q = e^{2i\pi\tau}$ and $z = e^{2i\pi\nu}$, with $\tau, \nu \in \mathbb{C}$ and $Im(\tau) > 0$ in order for the character series to be convergent. The twisted Ramond characters may be obtained from the untwisted ones by spectral flow with *integer twist* θ. Note that we twist by $-\theta$ (for historical reasons) in the following definition. Since

$$Tr(q^{[\mathcal{L}_0 - \theta\mathcal{J}_0 + \frac{c}{6}(\theta^2 - \theta)]} z^{[\mathcal{J}_0 - \frac{c}{3}\theta]}) = z^{-\frac{c}{3}\theta} q^{\frac{c}{6}(\theta^2 - \theta)} Tr(q^{\mathcal{L}_0}(zq^{-\theta})^{\mathcal{J}_0}), \qquad (2.21)$$

we have,

$$X^{N=2}_{r,s,u,p;\theta}(z, q) = z^{-\frac{c}{3}\theta} q^{\frac{c}{6}(\theta^2 - \theta)} X^{N=2}_{r,s,u,p}(zq^{-\theta}, q), \qquad (2.22)$$

where the untwisted $N = 2$ Ramond characters (2.20) read,

$$X^{N=2}_{r,s,u,p}(z,q) = z^{s-1-\frac{p}{u}(r-1)} \frac{\vartheta_{1,0}(z,q)}{\eta(q)^3} \phi_{r,s,u,p}(z,q), \tag{2.23}$$

with

$$\phi_{r,s,u,p}(z,q) = \sum_{m\in\mathbb{Z}} q^{m^2up-mu(s-1)} \left(\frac{q^{mpr}}{1+z^{-1}q^{mu}} - q^{r(s-1)} \frac{q^{-mpr}}{1+z^{-1}q^{mu-r}} \right). \tag{2.24}$$

We use the theta functions,

$$\begin{aligned}
\vartheta_{1,0}(z,q) &= z^{-\frac{1}{2}} \sum_n q^{\frac{1}{2}(n+\frac{1}{2})^2} z^{(n+\frac{1}{2})} = q^{1/8} \prod_{m\geq 1} (1+z^{-1}q^{m-1})(1+zq^m)(1-q^m), \\
\vartheta_{1,1}(z,q) &= -iz^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} (-1)^{n+\frac{1}{2}} q^{\frac{1}{2}(n+\frac{1}{2})^2} z^{(n+\frac{1}{2})} \\
&= q^{1/8} \prod_{m\geq 1} (1-z^{-1}q^{m-1})(1-zq^m)(1-q^m)
\end{aligned} \tag{2.25}$$

and the Dedekind function

$$\eta(q) = q^{\frac{1}{24}} \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}(3n^2+n)} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n). \tag{2.26}$$

The range (2.18) for the parameters r, s, θ labelling the admissible characters (2.22) results from the following properties. First of all, for $\alpha \in \mathbb{Z}$, one has,

$$X^{N=2}_{\alpha u,s,u,p;\theta}(z,q) = 0 \tag{2.27}$$

and

$$X^{N=2}_{\alpha u+r,s,u,p;\theta}(z,q) = X^{N=2}_{r,s-\alpha p,u,p;\theta}(z,q). \tag{2.28}$$

When α is even, $X^{N=2}_{2ku+r,s,u,p;\theta}(z,q) = X^{N=2}_{r,s-2kp,u,p;\theta}(z,q) = X^{N=2}_{r,s,u,p;\theta-ku}(z,q)$. When α is odd however, the characters satisfy,

$$\begin{aligned}
X^{N=2}_{(2k+1)u+r,s,u,p;\theta}(z,q) &= X^{N=2}_{r,s-(2k+1)p,u,p;\theta}(z,q) = X^{N=2}_{u+r,s,u,p;\theta-ku}(z,q) \\
&= X^{N=2}_{r,s-p,u,p;\theta-ku}(z,q).
\end{aligned} \tag{2.29}$$

Second of all, the function $\phi_{r,s,u,p;\theta}(z,q) \equiv \phi_{r,s,u,p}(zq^{-\theta},q)$ *is not periodic in the spectral flow parameter θ for p other than one.* Indeed, using the explicit expression (2.24), one can show that, for any integer $n \geq 1$, one has ([57]),

$$\begin{aligned}
&z^{2pn} q^{pn(r-un)-un(s-1)} \phi_{r,s,u,p}(zq^{-un},q) - \phi_{r,s,u,p}(z,q) = \\
&\sum_{a=0}^{2pn-1} (-1)^{a+1} z^{a+1} q^{-\frac{up}{4}} [\vartheta_{1,0}(q^{-up+u(s+a)-pr}, q^{2up}) - q^{r(s+a)} \vartheta_{1,0}(q^{-up+u(s+a)+pr}, q^{2up})]
\end{aligned}$$

(2.30)

while for any integer $n \leq -1$,

$$z^{2pn}q^{pn(r-un)-un(s-1)}\phi_{r,s,u,p}(zq^{-un},q) - \phi_{r,s,u,p}(z,q) =$$
$$\sum_{a=2pn}^{-1}(-1)^a z^{a+1}q^{-\frac{up}{4}}[\vartheta_{1,0}(q^{-up+u(s+a)-pr},q^{2up}) - q^{r(s+a)}\vartheta_{1,0}(q^{-up+u(s+a)+pr},q^{2up}). \tag{2.31}$$

This non quasi-periodicity for p non unity is an obstruction to write the admissible characters in terms of products of theta functions, and is a source of difficulty when studying the modular properties of these characters as we shall discuss later.

We end up this section with a parenthesis on the special case $p = 1$, which corresponds to the well-known unitary minimal $N = 2$ characters at central charge $c = 3(1 - \frac{2}{u})$. We set $p = 1$ in (2.22) and write,

$$\begin{aligned} X^{N=2}_{r,s,u,1;\theta}(z,q) &= z^{-\frac{c}{3}\theta}q^{\frac{c}{6}(\theta^2-\theta)}X^{N=2}_{r,s,u,1}(zq^{-\theta},q) \\ &= z^{-\frac{c}{3}\theta}q^{\frac{c}{6}(\theta^2-\theta)}[zq^{-\theta}]^{s-1-\frac{1}{u}(r-1)}\frac{\theta_{1,0}(zq^{-\theta},q)}{\eta^3(q)}\phi_{r,s,u,1}(zq^{-\theta},q), \end{aligned} \tag{2.32}$$

with

$$\phi_{r,s,u,1}(zq^{-\theta},q) = \sum_{m\in\mathbb{Z}} q^{um^2-mu(s-1)}\left[\frac{q^{mr}}{1+z^{-1}q^{mu+\theta}} - q^{r(s-1)}\frac{q^{-mr}}{1+z^{-1}q^{mu+\theta-r}}\right]. \tag{2.33}$$

A remarkable property of the above series is that it can be rewritten as the following infinite product, for the two allowed values $s = 0, 1$ in the range (2.18). One in fact has,

$$\begin{aligned} &\phi_{r,1,u,1}(zq^{-\theta},q) = -z^{-1}q^{\theta}\phi_{r,0,u,1}(zq^{-\theta},q) \\ = &\frac{\prod_{n=1}^{\infty}(1-q^{u(n-1)+r})(1-q^{un-r})(1-q^{un})^2}{\prod_{n=1}^{\infty}(1+zq^{un-\theta})(1+z^{-1}q^{u(n-1)+\theta})(1+z^{-1}q^{un-r+\theta})(1+zq^{u(n-1)+r-\theta})}, \end{aligned} \tag{2.34}$$

as can be checked by a residue analysis (see [59]). This in turn leads to,

$$X^{N=2}_{r,1,u,1;\theta}(z,q) = -X^{N=2}_{r,0,u,1;\theta}(z,q), \tag{2.35}$$

and to the conclusion the parameter s can be chosen to be $s = 1$ when $p = 1$. Using the Jacobi triple product identity

$$\sum_{n\in z} s^{n^2}t^n = \prod_{n=1}^{\infty}(1-s^{2n})(1+ts^{2n-1}t)(1+ts^{2n-1}t^{-1}) \tag{2.36}$$

and the functions (2.25), we rewrite (2.34) as,

$$\phi_{r,1,u,1}(zq^{-\theta}, q) = \frac{\eta^3(q^u)\vartheta_{1,1}(q^{-r}, q^u)}{\vartheta_{1,0}(zq^{-\theta}, q^u)\vartheta_{1,0}(z^{-1}q^{\theta-r}, q^u)} = -\frac{\eta^3(q^u)\vartheta_{1,1}(q^{r}, q^u)}{\vartheta_{1,0}(zq^{r-\theta}, q^u)\vartheta_{1,0}(z^{-1}q^{\theta}, q^u)}. \tag{2.37}$$

To summarize, the unitary $N = 2$ Ramond characters at central charge $c = 3(1-\frac{2}{u})$, $u \in \mathbb{N} \setminus \{1\}$ are given by,

$$\begin{aligned} X^{N=2}_{r,1,u,1;\theta}(z,q) &= X^{N=2}_{r,u;\theta}(z,q) \\ &= z^{-c/6} z^{\frac{1}{2}} z^{\frac{2\theta-r}{u}} q^{\frac{\theta(r-\theta)}{u}} \frac{\vartheta_{1,0}(z,q)}{\eta^3(q)} \frac{\eta^3(q^u)\vartheta_{1,1}(q^{-r}, q^u)}{\vartheta_{1,0}(zq^{-\theta}, q^u)\vartheta_{1,0}(z^{-1}q^{\theta-r}, q^u)} \\ &= -z^{-c/6} z^{\frac{1}{2}} z^{\frac{2\theta-r}{u}} q^{\frac{\theta(r-\theta)}{u}} \frac{\vartheta_{1,0}(z,q)}{\eta^3(q)} \frac{\eta^3(q^u)\vartheta_{1,1}(q^{r}, q^u)}{\vartheta_{1,0}(zq^{r-\theta}, q^u)\vartheta_{1,0}(z^{-1}q^{\theta}, q^u)}. \end{aligned} \tag{2.38}$$

Let us discuss the range (2.18) further. It is a well-known fact that there are $\frac{u(u-1)}{2}$ irreducible unitary $N = 2$ characters at central charge $c = 3(1 - \frac{2}{u})$. A priori, the range (2.18) when $p = 1$ allows θ to be any integer. However the characters (2.38) satisfy two remarkable properties, namely,

$$X^{N=2}_{r,u;\theta}(z,q) = X^{N=2}_{r,u;\theta+u}(z,q) \tag{2.39}$$

$$\text{and} \quad X^{N=2}_{r,u;\theta+r}(z,q) = X^{N=2}_{u-r,u;\theta}(z,q), \tag{2.40}$$

so that the fundamental range (2.18) is effectively restricted to $1 \leq r, \theta \leq u-1$ and $\theta \leq r$.

Note that we have adopted the conventions of [57]. They differ very slightly from the conventions in [41] and [42] by a factor $z^{-c/6}$. One has,

$$X^{N=2}_{r,u;\theta}(z,q) = z^{-c/6} Ch^{R,N=2}_{\theta,r-\theta}(z,q), \tag{2.41}$$

where $Ch^{R,N=2}_{\theta,r-\theta}(z,q)$ are the Ramond irreducible unitary $N = 2$ characters as they appear in [41]. This is easily established once the parameter θ is relabelled j and $r - \theta$ is relabelled k, while $\phi_{r,1,u,1}(zq^{-\theta}, q)$ is relabelled $\Gamma^{(u)}_{\theta,r-\theta}(z,q)$.

We stress once more that the periodicity property (2.39), which is intimately related to the fact the characters can be written in the form of products of theta functions and therefore have a standard behaviour under the modular group, *does not survive when p is different from one.* The main object of this chapter is to study how this non-periodicity for higher values of the parameter p influences the behaviour of minimal non-unitary $N = 2$ characters under the modular group.

2.4 Modular transformations of irreducible $N = 2$ minimal characters

A conformal field theory must satisfy several properties if it is to be of use in string theory. One of them is modular invariance. Let us review how modular invariance severely constrains the theory of closed bosonic strings. The vacuum-to-vacuum amplitude for interacting, closed bosonic strings is given by the Polyakov path integral,

$$Z = \sum_{\text{genus } h=0}^{\infty} (g_s)^{2h-2} \int \mathcal{D}^D X^\mu \mathcal{D} g_{\alpha\beta} e^{-S(X,g)}, \quad \alpha, \beta = 1, 2; \mu = 0, .., D-1, \quad (2.42)$$

where one sums over all two-dimensional surfaces swept by closed strings in interaction, organising them by their genus h, and weighting each term by the strength of the interaction, encoded in an appropriate power of the string coupling constant g_s. The string action is given by,

$$S(X, g) = -\frac{1}{2} T \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \quad (2.43)$$

and the $g_{\alpha\beta}$ integral is over the intrinsic shapes of two-dimensional surfaces at fixed genus. The X^μ integral is over the different ways of embedding a two-dimensional surface in D-dimensional space-time. The vacuum-to-vacuum contribution of a free closed string evolving in a higher D-dimensional space corresponds to an infinitely long cylindrical worldsheet , and therefore, the leading term in (2.42) is a sphere (genus zero) and it gives a classical tree-level contribution to the amplitude. Any non-perturbative contributions like instantons for instance are missing from the Polyakov integral, and would typically be proportional to e^{-1/g_s^2} or e^{-1/g_s}. We will not discuss them here.

The Polyakov integral can be generalised to include external states by introducing vertex operators. It requires correlation functions to be well-defined on any closed two-dimensional surface, and different parametrisations of the same two-dimensional surface should give the same answer. Surfaces of genus $h \geq 1$ have a set of complex parameters called moduli, whose values are modified by large reparametrisations (i.e. not continuously connected to the identity) called modular transformations. These transformations do not change the shape of the surface, and any correlation function on the surface should therefore be invariant under them. The lowest genus surface with a modulus is the torus ($h = 1$) (actually, the torus has exactly one modulus traditionally called τ, while a surface of genus $h > 1$ has $3(h-1)$ moduli). The vacuum amplitude on the torus is called the partition function of the theory, and it describes the one-loop quantum correction of a free closed string. Its modular invariance is a very strong constraint on the theory.

A two-dimensional torus is constructed by identifying points of the complex z-plane in the following way,

$$z \equiv z + \omega_1, \qquad z \equiv z + \omega_2, \tag{2.44}$$

with ω_1 and ω_2 the two periods, whose ratio is the modulus, i.e. $\omega_1/\omega_2 = \tau$. The most general basis to describe the above torus is actually $(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, so that the modular transformations of the torus are

$$F_{a,b,c,d} : \tau \to \frac{a\tau + b}{c\tau + d} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1.$$

Since $F_{a,b,c,d} = F_{-a,-b,-c,-d}$, the group of modular transformations on the torus is actually $PSL(2, \mathbb{Z})$ rather than $SL(2, \mathbb{Z})$, the former being the quotient of the latter by the discrete subgroup $\{I, -I\}$, with I the identity. The group has two generators conventionally denoted S and T, with matrix representation,

$$\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and obeying the relations $S^2 = I = (ST)^3$.

The partition function on a torus with periods ω_2 (defining a cylinder) and ω_1 (defining a time period for the time coordinate running along the cylinder) is,

$$Z(q) = Tr(e^{\omega_2 {L_{-1}}^{cyl} + \bar{\omega}_2 \bar{L}_{-1}^{cyl}}) = Tr(q^{{L_0}^{plane} - \frac{c}{24}} \bar{q}^{\bar{L}_0^{plane} - \frac{\bar{c}}{24}}),$$

where $q = exp(2\pi i \frac{\omega_1}{\omega_2})$ and we used the fact that under a conformal transformation from the cylinder (with period ω_1) to the plane we have

$$L_1{}^{cyl} = \frac{2\pi i}{\omega_1}(L_0{}^{plane} - \frac{c}{24}).$$

We recall that this construction gives a very useful relation between the path integral of a CFT on the torus and a trace over the Hilbert space. First, the propagation along the cylinder is governed by the Hamiltonian

$$H = \frac{2\pi}{\omega_1}(L_0 + \bar{L}_0 - \frac{c}{12}), \tag{2.45}$$

and rotations around the cylider are implemented by the momentum operator

$$P = \frac{2\pi}{\omega_1}(L_0 - \bar{L}_0 - \frac{\bar{c}}{12}), \tag{2.46}$$

where $c = \bar{c}$. H and P are generating transformations along the time and space directions. Conformal invariance tells us that the Hilbert space splits as a sum of representations of the conformal algebra and affine Kac - Moody algebra. So accordingly in the presence of conformal symmetry alone, the torus partition function has the following form,

$$Z(q) = \sum_{h,\bar{h}} X_h(q) N_{h,\bar{h}} \bar{X}_{\bar{h}}(\bar{q}), \tag{2.47}$$

where X_h are characters for the irreducible representations of the Virasoro algebra with highest weight of conformal dimension h.

In the presence of a Kac - Moody symmetry, the partition is given by,

$$Z(q, \rho_i) = Tr_H(q^{L_0 - \frac{c}{24}} e^{2\pi i \sum_i \rho_i H_0^i} \bar{q}^{\bar{L}_0 - \frac{c}{24}} e^{2\pi i \sum_i \bar{\rho}_i \bar{H}_0^i}). \tag{2.48}$$

where H_0^i are the zero modes of the Cartan generators.

This trace is also over the full Hilbert space of states of the theory, and may be decomposed into a trace over irreducible modules. We have

$$Z(q, \rho_i) = \sum_{\lambda,\nu} X_\lambda(q, \rho_i) N_{\lambda,\nu} \bar{X}_\nu(\bar{q}, \bar{\rho}_i), \tag{2.49}$$

where $X_\lambda(q, \rho_i) = Tr_H(q^{L_0 - \frac{c}{24}} e^{2\pi i \sum_i \rho_i H_0^i})$ are characters for the affine algebra $\hat{g}$. So the partition function will be expressible as a bilinear combination of characters.

In a two-dimensional conformal field theory with bosons and fermions, one must take into account the number of topologically distinct ways to put a spinor field on a genus g surface, in other words, the number of spin structures on that surface. In superspace, a torus can be described as the (z, θ) plane (θ here is a Grassmann variable) modded out by a group of global superconformal transformations,

$$(z, \theta) \equiv (z + \omega_1, \eta_1 \theta) \equiv (z + \omega_2, \eta_2 \theta), \tag{2.50}$$

where $\eta_1, \eta_2 = \pm 1$ define the four spin structures of the torus. The choice $(\eta_1, \eta_2) = (-1, -1)$ corresponds to the Neveu-Schwarz (NS) sector of the theory, while the choice $(\eta_1, \eta_2) = (1, -1)$ corresponds to the Ramond (R) sector. Furthermore, $(\eta_1, \eta_2) = (-1, 1)$ is the super NS sector and $(\eta_1, \eta_2) = (1, 1)$ is the super R sector. The building blocks of the partition function are the characters of irreducible representations of the infinite dimensional symmetry algebra and they are of NS, R, super NS and super R types. Under the S transformation, which basically interchanges the two periods ω_1 and ω_2, the Ramond characters transform into super NS characters, while the NS characters transform into NS characters and the super R transform into super R characters. In order to construct modular invariant partition functions, it is crucial to know how the characters transform under the modular group. The behaviour under T is straightforward while the S transform is usually much more complicated. In the case of unitary minimal $N = 2$ characters, this behaviour has been known for a long time because of the relevance of unitary representations in the description of $N = 2$ superstrings. We will however rederive the S transform of the Ramond unitary minimal characters as a warm up exercise before tackling the much more involved case of non unitary minimal characters, whose relevance in a physical context has been the object of constant debate over the last decade. Nevertheless, their study is a mathematical challenge we have taken up.

In $N = 2$ superconformal symmetry, recall the characters are functions of the complex modulus τ and of a complex angle ν, and the modular transformations S and T act on these variables as,

$$S(\nu, \tau) = (\frac{\nu}{\tau}, -\frac{1}{\tau}) \qquad T(\nu, \tau) = (\nu, \tau + 1). \tag{2.51}$$

In the previous section, we described irreducible characters in the R sector. To obtain the NS characters, one allows the spectral flow parameter θ to be half integer. Typically, the NS characters are given by,

$$\begin{aligned} X^{N=2}_{r,s,u,p;\theta\pm\frac{1}{2}}(z,q) &= z^{-\frac{c}{3}(\theta\pm\frac{1}{2})}q^{\frac{c}{6}[(\theta\pm\frac{1}{2})^2-(\theta\pm\frac{1}{2})]}X^{N=2}_{r,s,u,p}(zq^{-\theta\mp\frac{1}{2}},q) \\ &= z^{\mp\frac{c}{6}}q^{\frac{c}{6}[\frac{1}{4}\mp\frac{1}{2}]}X^{N=2}_{r,s,u,p;\theta}(zq^{\mp\frac{1}{2}},q), \end{aligned} \tag{2.52}$$

where we have used (2.22). We will choose to twist by $\theta-\frac{1}{2}$ in the following, as comparison with [41] and [42] for $p=1$ is straightforward in that case. The super-characters are obtained by inserting the operator $(-1)^F$ in the trace (2.20), with F the fermion number. This is equivalent to evaluate the R and NS characters at the variable $-z$ or $\nu+\frac{1}{2}$. Hence, the R and NS super-characters are respectively given by,

$$X^{N=2}_{r,s,u,p;\theta}(-z,q) \quad \text{and} \quad X^{N=2}_{r,s,u,p;\theta\pm\frac{1}{2}}(-z,q). \tag{2.53}$$

Consider thus the unitary minimal characters (2.38) where $q=e^{2i\pi\tau}$ and $z=e^{2i\pi\nu}$, and introduce the function

$$C(\mu,\nu,\tau) = \frac{\eta(\tau)^3\vartheta_{1,1}(\nu-\mu,\tau)}{\vartheta_{1,0}(-\mu,\tau)\vartheta_{1,0}(\nu,\tau)}, \tag{2.54}$$

so that

$$X^{N=2}_{r,u;\theta}(\nu,\tau) = -e^{2i\pi[\frac{2\theta-r+1}{u}]\nu}e^{2i\pi[\frac{\theta(r-\theta)}{u}]\tau}\frac{\vartheta_{1,0}(\nu,\tau)}{\eta(\tau)^3}C(\nu-\theta\tau,\nu+(r-\theta)\tau,u\tau). \tag{2.55}$$

Under S, we expect them to transform as a linear combination of super-NS characters of the type,

$$\begin{aligned} X^{N=2}_{r,u;\theta-\frac{1}{2}}(\nu+\frac{1}{2},\tau) &= -e^{i\pi[\frac{1}{2}+\frac{2\theta-r}{u}]}z^{\frac{1}{2}+\frac{2\theta-r}{u}}q^{\frac{c}{8}+\frac{2\theta-r+1}{2u}+\frac{\theta(r-\theta)}{u}}\frac{\vartheta_{1,0}(\nu+\frac{\tau}{2}+\frac{1}{2},\tau)}{\eta(\tau)^3} \\ &\quad\times C(\nu-(\theta-\frac{1}{2})\tau+\frac{1}{2},\nu-(\theta-r-\frac{1}{2})\tau+\frac{1}{2},u\tau). \end{aligned} \tag{2.56}$$

Let us first calculate the S transform of $C(\nu-\theta\tau,\nu-(\theta-r)\tau,u\tau)$. The theta functions (2.25) and the Dedekind functions transform as,

$$\begin{aligned} \vartheta_{1,0}(\frac{\nu}{\tau},\frac{-1}{\tau}) &= (-i\tau)^{\frac{1}{2}}e^{-\frac{i\pi\tau}{4}}e^{i\pi\frac{\nu^2-\nu}{\tau}}\vartheta_{1,0}(\nu-\frac{\tau}{2}+\frac{1}{2},\tau), \\ &= -(-i\tau)^{\frac{1}{2}}e^{\frac{3i\pi\tau}{4}}e^{i\pi\frac{\nu^2-\nu}{\tau}}e^{2i\pi\nu}\vartheta_{1,0}(\nu+\frac{\tau}{2}+\frac{1}{2},\tau), \\ \vartheta_{1,1}(\frac{\nu}{\tau},\frac{-1}{\tau}) &= -i(-i\tau)^{\frac{1}{2}}e^{i\pi(\frac{\nu^2-\nu}{\tau}+\nu)}\vartheta_{1,1}(\nu,\tau), \\ \eta(-\frac{1}{\tau}) &= (-i\tau)^{\frac{1}{2}}\eta(\tau), \end{aligned} \tag{2.57}$$

and this leads to,

$$C(\frac{\mu}{\tau}, \frac{\nu}{\tau}, \frac{-1}{\tau}) = \frac{\eta(\frac{-1}{\tau})^3 \vartheta_{1,1}(\frac{\nu-\mu}{\tau}, \frac{-1}{\tau})}{\vartheta_{1,0}(-\frac{\mu}{\tau}, \frac{-1}{\tau})\vartheta_{1,0}(\frac{\nu}{\tau}, \frac{-1}{\tau})} \tag{2.58}$$

$$= \tau e^{-i\pi(\nu+\mu) - \frac{\pi i \tau}{2} - 2\pi i \frac{\mu\nu}{\tau}} C(\mu + \frac{\tau}{2} + \frac{1}{2}, \nu + \frac{\tau}{2} + \frac{1}{2}, \tau), \tag{2.59}$$

so that the S transform of $C(\nu - \theta\tau, \nu + (r - \theta)\tau, u\tau)$ is,

$$C(\frac{\nu+\theta}{\tau}, \frac{\nu + (-r+\theta)}{\tau}, \frac{-u}{\tau}) =$$
$$\frac{\tau}{u} e^{-i\pi\frac{\tau}{2u} + 2i\pi[-\frac{\nu+\theta}{u} + \frac{r}{2u} + r\frac{(\nu+\theta)}{u\tau} - \frac{(\nu+\theta)^2}{u\tau}]} C(\frac{\nu+\theta}{u} + \frac{\tau}{2u} + \frac{1}{2}, \frac{\nu+\theta-r}{u} + \frac{\tau}{2u} + \frac{1}{2}, \frac{\tau}{u}), \tag{2.60}$$

and it follows that the S transform of (2.55) is,

$$X^{N=2}_{r,u;\theta}(\frac{\nu}{\tau}, -\frac{1}{\tau}) = \frac{i}{u} e^{-\pi i \frac{2\nu(1-u)+2\theta-r}{u}} e^{\frac{i\pi}{\tau}(1-\frac{2}{u})\nu(\nu-1)} e^{\frac{i\pi\tau}{4}(3-\frac{2}{u})} \frac{\vartheta_{1,0}(\nu + \frac{\tau}{2} + \frac{1}{2}, \tau)}{\eta(\tau)^3}$$
$$\times C(\frac{\nu+\theta}{u} + \frac{\tau}{2u} + \frac{1}{2}, \frac{\nu+\theta-r}{u} + \frac{\tau}{2u} + \frac{1}{2}, \frac{\tau}{u}). \tag{2.61}$$

If we want to interpret the right hand side as a combination of $N = 2$ characters, the C function should have $u\tau$ as its third argument instead of $\frac{\tau}{u}$. Note one can write ,

$$C(\alpha, \beta, \frac{\tau}{u}) = \sum_{a=0}^{u-1}\sum_{b=0}^{u-1} e^{2i\pi[b\alpha - a\beta] + i\pi[a+b] + \frac{2\pi i a b \tau}{u}}$$
$$\times C(u\alpha + \tau a + \frac{u+1}{2}, u\beta - \tau b + \frac{u+1}{2}, u\tau)$$
$$\equiv \sum_{a=0}^{u-1}\sum_{b=0}^{u-1} f(a,b), \tag{2.62}$$

where

$$\sum_{a=0}^{u-1}\sum_{b=0}^{u-1} f(a,b) = \sum_{r=1}^{u-1}\sum_{a=0}^{r} f(a, r-a) + \sum_{r=1}^{u-1}\sum_{a=r+1}^{u-1} f(a, u+r-a). \tag{2.63}$$

We recall that (2. 64) only is true for the specific function given in (2. 63). Relabelling $r \to u - r$ and $a \to u - a$, we arrive at ,

$$\sum_{a=0}^{u-1}\sum_{b=0}^{u-1} f(a,b) = \sum_{r=1}^{u-1}\sum_{a=r}^{u} f(u-a, a-r) + \sum_{r=1}^{u-1}\sum_{a=1}^{r-1} f(u-a, a+u-r), \tag{2.64}$$

and finally,

$$\sum_{a=0}^{u-1}\sum_{b=0}^{u-1} f(a,b) = \sum_{r=1}^{u-1}\sum_{a=1}^{u} f(-a, a-r). \tag{2.65}$$

Let us now apply this trick to the function C as it appears in (2.61), and consider

$$\begin{aligned} \alpha &= \frac{\nu+\theta}{u} + \frac{\tau}{2u} + \frac{1}{2} \\ \text{and} \quad \beta &= \frac{\nu+\theta-r}{u} + \frac{\tau}{2u} + \frac{1}{2} \end{aligned} \tag{2.66}$$

in (2.62). We find,

$$\begin{aligned} X^{N=2}_{r,u;\theta}(\frac{\nu}{\tau}, -\frac{1}{\tau}) = \frac{i}{u} e^{-i\pi\frac{2\nu(1-u)+2\theta-r}{u}} e^{\frac{i\pi}{\tau}(1-\frac{2}{u})\nu(\nu-1)} e^{\frac{i\pi\tau}{4}(3-\frac{2}{u})} \frac{\vartheta_{1,0}(\nu+\frac{\tau}{2}+\frac{1}{2},\tau)}{\eta(\tau)^3} \\ \times \sum_{r'=1}^{u-1}\sum_{a=1}^{u} f(-a, a-r'), \end{aligned} \tag{2.67}$$

where

$$\begin{aligned} \sum_{r'=1}^{u-1}\sum_{a=1}^{u} f(-a,a-r') = \sum_{r'=1}^{u-1}\sum_{a=1}^{u} e^{\frac{i\pi}{u}[(a-r')(2\theta+2\nu+\tau)+a(2\theta+2\nu-2r+\tau)-2a(a-r')\tau]} \\ \times C(\nu-(a-\frac{1}{2})\tau+\frac{1}{2}, \nu-(a-r'-\frac{1}{2})\tau+\frac{1}{2}, u\tau), \end{aligned} \tag{2.68}$$

since

$$\begin{aligned} C(u\alpha - \tau a + \frac{u+1}{2}, u\beta - \tau(a-r') + \frac{u+1}{2}, u\tau) = \\ C(\nu-(a-\frac{1}{2})\tau+\frac{1}{2}+\theta+u, \nu-(a-r'-\frac{1}{2})\tau+\frac{1}{2}+\theta-r+u, u\tau) = \\ C(\nu-(a-\frac{1}{2})\tau+\frac{1}{2}, \nu-(a-r'-\frac{1}{2})\tau+\frac{1}{2}, u\tau). \end{aligned} \tag{2.69}$$

Now compare with (2.56). Put $\theta = a, r = r'$ in that formula to rewrite,

$$\begin{aligned} X^{N=2}_{r,u;\theta}(\frac{\nu}{\tau}, -\frac{1}{\tau}) = -\frac{1}{u} e^{i\pi(1-\frac{2}{u})(\frac{\nu^2-\nu}{\tau}+\nu)} \\ \times \sum_{r'=1}^{u-1}\sum_{a=1}^{u} e^{-i\pi\frac{rr'+1}{u}+\frac{i\pi}{u}(2\theta-r-1)(2a-r'-1)} X^{N=2}_{r',u;a-\frac{1}{2}}(\nu+\frac{1}{2},\tau). \end{aligned} \tag{2.70}$$

Finally, we can exploit the isomorphisms (2.39) and (2.40) to express the above modular transformation property in a form directly comparable with the literature [41] and [42]. Changing the summation $\sum_{r'=1}^{u-1}\sum_{a=1}^{u}$ in (2.70) by,

$$\sum_{r'=1}^{u-1}\sum_{a=0}^{u-1}=\sum_{r'=1}^{u-1}(\sum_{a=0}^{r'-1}+\sum_{a=r'}^{u-1}), \tag{2.71}$$

and relabelling $r'=u-r''$, $a=a'-r''+u$ in the second term, one obtains the following S transformation behaviour for the unitary $N=2$ superconformal characters,

$$\begin{aligned}\chi_{r,u;\theta}^{N=2}(\frac{\nu}{\tau},-\frac{1}{\tau})=&\frac{2}{u}e^{i\pi(1-\frac{2}{u})(\frac{\nu^2-\nu}{\tau}+\nu+\frac{1}{2})}\\&\sum_{r'=1}^{u-1}\sum_{a=0}^{r'-1}e^{\frac{i\pi}{u}(2\theta-r-1)(2a-r'-1)}\sin\frac{\pi rr'}{u}\chi_{r',u;a-\frac{1}{2}}^{N=2}(\nu+\frac{1}{2},\tau).\end{aligned} \tag{2.72}$$

The derivation of how non-unitary minimal $N=2$ characters (2.22) transform under S is more involved, and as we stressed before, can be traced to the fact that these characters are not periodic under the spectral flow (2.30) (2.31). We will use a result by [57], namely, that the affine $\widehat{sl(2)}$ characters can be branched into $N=2$ characters. This is a consequence of the equivalence between categories of representations of the $\widehat{sl(2)}$ and $N=2$ algebras [60].

The affine $\widehat{sl(2)}$ algebra is defined by the commutation relations

$$\begin{aligned}&[J_m^3,J_n^{\pm}]=\pm J_{m+n}^{\pm},\quad [J_m^3,J_n^3]=\frac{k}{2}m\delta_{m+n,0}\\&[J_m^+,J_n^-]=2J_{m+n}^3+km\delta_{m+n,0},\end{aligned} \tag{2.73}$$

and is (in particular) isomorphic under the spectral flow transformations,

$$J_n'^{\pm}=J_{n\pm\theta}^{\pm}\ ,\ J_n'^3=J_n^3+\frac{k}{2}\theta\delta_{n,0}, \tag{2.74}$$

where the twist $\theta\in\mathbb{Z}$. Note that under the spectral flow, the Sugawara energy-momentum tensor transforms as,

$$L_n'=L_n+\theta J_n^3+\frac{k}{4}\theta^2\delta_{n,0}. \tag{2.75}$$

The character of an irreducible highest weight state representation V is formally given by,

$$\chi_V^{\widehat{sl(2)}}(z,q)=Tr_V(q^{L_0-c/24}z^{J_0^3}) \tag{2.76}$$

where $c = \frac{3k}{k+2}$ is the central charge of the associated Virasoro algebra. The variables z, q are complex, with $|q| < 1$.

We need to consider here the branching of admissible $\widehat{sl(2)}$ characters at level $k = \frac{u}{p} - 2$, with $\frac{u}{p} > 0$ and u and p coprime. These characters have been thoroughly studied [61,62]. The *untwisted* characters are labelled by two integers r and s in the ranges $1 \leq r \leq u-1$ and $1 \leq s \leq p$, and given by,

$$X^{\widehat{sl(2)}}_{r,s,u,p}(z,q) = \frac{\theta_{b_+,up}(z^{\frac{1}{p}},q) - \theta_{b_-,up}(z^{\frac{1}{p}},q)}{\theta_{1,2}(z,q) - \theta_{-1,2}(z,q)}, \tag{2.77}$$

with $b_\pm = \pm pr - (s-1)u$ and the generalised theta functions given by $\theta_{m,\ell}(z,q) = \sum_{n\in\mathbb{Z}} q^{\ell(n+\frac{m}{2\ell})^2} z^{\ell(n+\frac{m}{2\ell})}$. Note the relation

$$X^{\widehat{sl(2)}}_{r,s-p,u,p}(z,q) = -X^{\widehat{sl(2)}}_{u-r,s,u,p}(z,q) \qquad \text{for } 1 \leq s \leq p \tag{2.78}$$

which will be extensively used in what follows.

Equivalently, in view of the definitions (2.25), one has,

$$X^{\widehat{sl(2)}}_{r,s,u,p}(z,q) = z^{\frac{r-1}{2}-(s-1)\frac{u}{2p}} q^{\frac{pr^2}{4u}-\frac{r}{2}(s-1)+\frac{u(s-1)^2}{4p}-\frac{up}{4}}$$
$$\frac{\vartheta_{1,0}(z^u q^{p(r-u)-(s-1)u}, q^{2up}) - z^{-r} q^{r(s-1)} \vartheta_{1,0}(z^u q^{-p(r+u)-(s-1)u}, q^{2up})}{\vartheta_{1,1}(z,q)}. \tag{2.79}$$

The branching of admissible $\widehat{sl(2)}$ into $N = 2$ characters occurs at a common central charge value of $c = \frac{3k}{k+2} = 3(1 - \frac{2p}{u})$. It is given, in the range $1 \leq r \leq u-1$ and $1 - p \leq s \leq p$, by,

$$X^{\widehat{sl(2)}}_{r,s,u,p}(z,q)\vartheta_{1,0}(zy,q) = \sum_{\theta\in\mathbb{Z}} X^{N=2}_{r,s,u,p;\theta}(y,q)\; y^{\frac{2p}{u}(j-\theta)} z^{j-\theta} q^{\frac{p}{u}(j-\theta+\frac{1}{2})^2}, \tag{2.80}$$

where the quantum number j is actually the $sl(2)$ isospin,

$$j = \frac{r-1}{2} - \frac{s-1}{2}\frac{u}{p}, \tag{2.81}$$

and the twisted $N = 2$ characters have been defined earlier (2.22).

The above sumrule provides us with an integral representation of the $N = 2$ non-unitary minimal characters. We write, for $n \in \mathbb{Z}$,

$$\frac{1}{2i\pi}\int_C dz z^{n-j} X^{\widehat{sl(2)}}_{r,s,u,p}(z,q)\vartheta_{1,0}(zy,q) =$$
$$\sum_{\theta\in\mathbb{Z}} X^{N=2}_{r,s,u,p;\theta}(y,q) y^{\frac{2p}{u}(j-\theta)} q^{\frac{p}{u}(j-\theta+\frac{1}{2})^2} \times \frac{1}{2i\pi}\int_C dz z^{n-\theta} =$$
$$X^{N=2}_{r,s,u,p;n+1}(y,q) y^{\frac{2p}{u}(j-n-1)} q^{\frac{p}{u}(j-n-\frac{1}{2})^2}. \tag{2.82}$$

We shall derive the S transform of untwisted non-unitary minimal $N=2$ characters for simplicity, that is we consider the above integral representation for $n=-1$. Setting $q=e^{2i\pi\tau}$, $z=e^{2i\pi\nu}$ and $y=e^{2i\pi\mu}$, we obtain,

$$X^{N=2}_{r,s,u,p}(\mu,\tau)=e^{-\frac{4i\pi}{u}\mu pj}e^{-\frac{2\pi i\tau}{u}p(j+\frac{1}{2})^2}\int_0^1 d\nu e^{-2i\pi\nu j}X^{\widehat{sl(2)}}_{r,s,u,p}(\nu,\tau)\vartheta_{1,0}(\mu+\nu,\tau). \quad (2.83)$$

Under S, these characters behave as,

$$\begin{aligned}X^{N=2}_{r,s,u,p}(\frac{\mu}{\tau},\frac{-1}{\tau})&=e^{-\frac{4i\pi}{u\tau}\mu pj+\frac{2i\pi}{u\tau}p(j+\frac{1}{2})^2}\int_0^1 d\nu e^{-2\pi i\nu j}X^{\widehat{sl(2)}}_{r,s,u,p}(\nu,\frac{-1}{\tau})\vartheta_{1,0}(\frac{\mu}{\tau}+\nu,\frac{-1}{\tau})\\&=e^{-\frac{4i\pi}{u\tau}\mu pj+\frac{2i\pi}{u\tau}p(j+\frac{1}{2})^2}\frac{1}{\tau}\int_0^\tau d\nu e^{-\frac{2i\pi}{\tau}\nu j}X^{\widehat{sl(2)}}_{r,s,u,p}(\frac{\nu}{\tau},\frac{-1}{\tau})\vartheta_{1,0}(\frac{\mu+\nu}{\tau},\frac{-1}{\tau}),\end{aligned} \quad (2.84)$$

where we have changed variable from ν to $\nu\tau$ in the second integral. We now use the well-known modular transformations of generalised theta functions to obtain those of affine $sl(2)$ admissible characters. In particular, the S modular transformation of the generalised theta functions is [63],

$$\theta_{m,\ell}(\nu/\tau,-1/\tau)=(-i\tau)^{\frac{1}{2}}(2\ell)^{-\frac{1}{2}}e^{\frac{i\pi\ell\nu^2}{2\tau}}\sum_{r\in\mathbb{Z}_{2\ell}}e^{-\frac{i\pi rm}{\ell}}\theta_{r,\ell}(\nu,\tau), \quad (2.85)$$

and leads to the following S transform of $\widehat{sl(2)}$ admissible characters,

$$\begin{aligned}X^{\widehat{sl(2)}}_{r,s,u,p}(\frac{\nu}{\tau},\frac{-1}{\tau})&=\frac{1}{2}\sqrt{\frac{2}{up}}e^{\frac{\pi i\nu^2}{\tau}(\frac{u}{2p}-1)}\\&\times\sum_{r'=1}^{u-1}\sum_{s'=1-p}^{p}e^{i\pi[r(s'-1)+r'(s-1)-(s-1)(s'-1)\frac{u}{p}]}\sin\frac{p\pi rr'}{u}X^{\widehat{sl(2)}}_{r',s',u,p}(\nu,\tau).\end{aligned} \quad (2.86)$$

We also use (2.57) to write,

$$\vartheta_{1,0}(\frac{\mu+\nu}{\tau},-\frac{1}{\tau})=-(-i\tau)^{\frac{1}{2}}e^{\frac{3i\pi\tau}{4}}e^{i\pi\frac{(\mu+\nu)^2-(\mu+\nu)}{\tau}}e^{2i\pi(\mu+\nu)}\vartheta_{1,0}(\mu+\nu+\frac{\tau}{2}+\frac{1}{2},\tau). \quad (2.87)$$

The expression (2.84) becomes,

$$\begin{aligned}X^{N=2}_{r,s,u,p}(\frac{\mu}{\tau},\frac{-1}{\tau})&=-\frac{1}{2}(-i)^{\frac{1}{2}}\tau^{-\frac{1}{2}}\sqrt{\frac{2}{up}}\\\times e^{-\frac{4i\pi}{u\tau}\mu pj+\frac{2i\pi}{u\tau}p(j+\frac{1}{2})^2}&e^{\frac{3i\pi\tau}{4}}e^{\frac{i\pi}{\tau}(\mu^2-\mu)}e^{2i\pi\mu}\sum_{r'=1}^{u-1}\sum_{s'=1-p}^{p}e^{i\pi[r(s'-1)+r'(s-1)-(s-1)(s'-1)\frac{u}{p}]}\sin\frac{p\pi rr'}{u}\\&\int_0^\tau d\nu e^{-\frac{2i\pi}{\tau}\nu(j+\frac{1}{2})}e^{\frac{i\pi}{2p\tau}\nu^2u}e^{\frac{2i\pi}{\tau}\mu\nu}e^{2i\pi\nu}X^{\widehat{sl(2)}}_{r',s',u,p}(\nu,\tau)\vartheta_{1,0}(\mu+\nu+\frac{\tau}{2}+\frac{1}{2},\tau).\end{aligned} \quad (2.88)$$

Now use the branching relation (2.80) as well as (2.52) to rewrite,

$$\begin{aligned}
&X^{\widehat{sl(2)}}_{r',s',u,p}(\nu,\tau)\vartheta_{1,0}(\mu+\nu+\frac{\tau}{2}+\frac{1}{2},\tau) = \\
&-i\sum_{\theta\in\mathbb{Z}} e^{\frac{2i\pi}{u}(2\mu+1)p(j'-\theta+\frac{1}{2})} e^{\frac{2i\pi\tau}{u}p(j'-\theta+1)^2} e^{2i\pi\nu(j'-\theta)-i\pi\mu} e^{-\frac{3i\pi\tau}{4}} X^{N=2}_{r',s',u,p;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau),
\end{aligned} \tag{2.89}$$

where

$$j' = \frac{(r'-1)}{2} - \frac{(s'-1)}{2}\frac{u}{p}. \tag{2.90}$$

Inserting the last relation in (2.88), and completing the squares in the ν-integral, we obtain,

$$\begin{aligned}
X^{N=2}_{r,s,u,p}(\frac{\mu}{\tau},\frac{-1}{\tau}) &= \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}} e^{i\pi(1-\frac{2p}{u})(\frac{\mu^2-\mu}{\tau}+\mu+\frac{1}{2})+i\pi s} \\
&\times \sum_{r'=1}^{u-1}\sum_{s'=1-p}^{p}\sum_{\theta\in\mathbb{Z}} e^{\frac{i\pi p}{u}(2\theta-r'-1)(-r-1)+i\pi s'} \sin\frac{p\pi r r'}{u} X^{N=2}_{r',s',u,p,;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau) \\
&\times L^{(u)}_{-2p\theta+p(r'+1)-u(s'-1)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})
\end{aligned} \tag{2.91}$$

where the integral $L^{(m)}_n(\tau,\eta)$ is defined as

$$L^{(m)}_n(\tau,\eta) = \int_{\eta+n\tau}^{\eta+(n+m)\tau} d\rho e^{i\pi\frac{\rho^2}{\tau}}. \tag{2.92}$$

This integral is defined by analytic continuation from the real axis, as the imaginary part of τ is positive. For $t\in\mathbb{R}_+$ and $y\in\mathbb{R}$, define the function

$$f^{(m)}_n(t,y) = \int_{y+nt}^{y+(n+m)t} dx e^{-\pi\frac{x^2}{t}}. \tag{2.93}$$

Continuing off the real axis, we have

$$L^{(m)}_n(\tau,\eta) = i f^{(m)}_n(-i\tau,-i\eta). \tag{2.94}$$

It is at this stage of the derivation that the complications due to the non-periodicity of $N=2$ non-unitary characters under the spectral flow emerge. Our first step is to exploit properties of the $N=2$ characters which derive from the fact they can be expressed in terms of generalised Appell functions whose study goes beyond the scope of this thesis. However, the following properties can be checked directly from the expressions presented in (2. 23).

For $s'-s>0$, one has,

$$X^{N=2}_{r',s',u,p,;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)=(-1)^{s'-s}X^{N=2}_{r',s,u,p,;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)$$
$$-iq^{-\frac{c}{24}-\frac{up}{4}+\frac{1}{2}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}e^{i\pi(s'-\frac{p}{u}(r'-2\theta))}y^{-\frac{1}{2}+s'-\frac{p}{u}(r'-2\theta)}\times$$
$$q^{-\frac{p}{u}(1-\theta)^2+(\frac{1}{2}-\theta)(s'-1-\frac{p}{u}(r'-1))}$$
$$\sum_{a=0}^{s'-s-1}y^{-a}q^{-(\frac{1}{2}-\theta)a}\Lambda_{s'-1-a,r'+1,u,p}(\tau),\quad(2.95)$$

while for $s'-s<0$, one has,

$$X^{N=2}_{r',s',u,p,;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)=(-1)^{s'-s}X^{N=2}_{r',s,u,p,;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)$$
$$+iq^{-\frac{c}{24}-\frac{up}{4}+\frac{1}{2}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}e^{i\pi(s'-\frac{p}{u}(r'-2\theta))}y^{-\frac{1}{2}+s'-\frac{p}{u}(r'-2\theta)}\times$$
$$q^{-\frac{p}{u}(1-\theta)^2+(\frac{1}{2}-\theta)(s'-1-\frac{p}{u}(r'-1))}$$
$$\sum_{a=s'-s}^{-1}y^{-a}q^{-(\frac{1}{2}-\theta)a}\Lambda_{s'-1-a,r'+1,u,p}(\tau),\quad(2.96)$$

where we have defined the function

$$\Lambda_{s,r+1,u,p}(\tau)=\vartheta_{1,0}(q^{-us+p(r-u)},q^{2up})-q^{rs}\vartheta_{1,0}(q^{-us-p(r+u)},q^{2up}).\quad(2.97)$$

Note for future reference that

$$\Lambda_{s,1,u,p}(\tau)=\Lambda_{s,u+1,u,p}(\tau)=0.\quad(2.98)$$

The formula (2.91) therefore becomes

$$X^{N=2}_{r,s,u,p}(\frac{\mu}{\tau},\frac{-1}{\tau})=\Psi^{(1)}+\Theta^{(1)}\quad(2.99)$$

with

$$\Psi^{(1)}=\frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi\frac{c}{6}}y^{\frac{c}{6}}$$
$$\times\sum_{r'=1}^{u-1}\sum_{\theta\in\mathbb{Z}}e^{\frac{i\pi p}{u}(2\theta-r'-1)(-r-1)}\sin\frac{p\pi rr'}{u}X^{N=2}_{r',s,u,p,;\theta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)$$
$$\times\sum_{s'=1-p}^{p}L^{(u)}_{-2p\theta+p(r'+1)-u(s'-1)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p}),\quad(2.100)$$

and

$$\Theta^{(1)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}} e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})} e^{i\pi s} y^{\frac{c}{6}+\frac{1}{2}} q^{-\frac{c}{24}-\frac{up}{4}+\frac{1}{2}} \frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}$$
$$\times \sum_{r'=1}^{u-1}\sum_{\theta\in\mathbb{Z}} e^{-\frac{i\pi p}{u}(2\theta-r'-1)r} y^{-\frac{p}{u}(r'-2\theta)} q^{-\frac{p}{u}(1-\theta)^2-\frac{p}{u}(\frac{1}{2}-\theta)(r'-1)} \sin\frac{p\pi r r'}{u}$$
$$\times \Big[\sum_{s'=s+1}^{p}\sum_{a=0}^{s'-s-1} - \sum_{s'=1-p}^{s-1}\sum_{a=s'-s}^{-1}\Big] y^{s'-1-a} q^{(\frac{1}{2}-\theta)(s'-1-a)} \Lambda_{s'-1-a,r'+1,u,p}(\tau)$$
$$\times L^{(u)}_{-2p\theta+p(r'+1)-u(s'-1)}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p}). \quad (2.101)$$

Let us first rearrange $\Theta^{(1)}$ by combining the summation on s' and a. Calling $a' = a - s'$, we see that the s' dependence disappears from all factors but $L^{(u)}_{-2p\theta+p(r'+1)-u(s'-1)}$. Note that

$$\sum_{s'=s+1}^{p}\sum_{a'=-s'}^{-s-1} = \sum_{a'=-s-1}^{-p}\sum_{s'=-a'}^{p} \quad \text{and} \quad \sum_{s'=1-p}^{s-1}\sum_{a'=-s}^{-s'-1} = \sum_{a'=-s}^{p-2}\sum_{s'=1-p}^{-a'-1}. \quad (2.102)$$

Moreover, as a direct consequence of the definition (2.92), one has,

$$\sum_{s'=-a'}^{p} L^{(u)}_{-2p\theta+p(r'+1)-u(s'-1)}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p}) =$$
$$L^{(u[a'+1+p])}_{-2p\theta+p(r'+1)+u(1-p)}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p}), \quad (2.103)$$

$$\sum_{s'=1-p}^{-a'-1} L^{(u)}_{-2p\theta+p(r'+1)-u(s'-1)}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p}) =$$
$$L^{(-u[a'+1-p])}_{-2p\theta+p(r'+1)+u(2+a')}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p}), \quad (2.104)$$

so that the expression for $\Theta^{(1)}$ becomes,

$$\Theta^{(1)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}} e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})} e^{i\pi s} y^{\frac{c}{6}+\frac{1}{2}} q^{-\frac{c}{24}-\frac{up}{4}+\frac{1}{2}} \frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}$$
$$\times \sum_{r'=1}^{u-1}\sum_{\theta\in\mathbb{Z}} e^{-\frac{i\pi p}{u}(2\theta-r'-1)r} y^{-\frac{p}{u}(r'-2\theta)} q^{-\frac{p}{u}(1-\theta)^2-\frac{p}{u}(\frac{1}{2}-\theta)(r'-1)} \sin\frac{p\pi r r'}{u}$$
$$\times \Big[\sum_{a'=-s-1}^{-p} L^{(u[a'+1+p])}_{-2p\theta+p(r'+1)+u(1-p)}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p}) -$$
$$\sum_{a'=-s}^{p-2} L^{(-u[a'+1-p])}_{-2p\theta+p(r'+1)+u(2+a')}(\frac{\tau}{2up}, \frac{2\mu-r}{2u}+\frac{s-1}{2p})\Big]$$
$$\times y^{-a'-1} q^{-(\frac{1}{2}-\theta)(a'+1)} \Lambda_{-a'-1,r'+1,u,p}(\tau). \quad (2.105)$$

A simpler expression can be provided, where the sum in the square bracket is replaced by a sum running from $s+1$ to $s+2p$. To see this, introduce the relabelling $a''=a'-2p$ in the second sum of the square bracket and use the easily derived property,

$$\Lambda_{-a''-1-2pj,r'+1,u,p}(\tau)=q^{-upj^2}q^{-u(a''+1)j-pr'j}\Lambda_{-a''-1,r'+1,u,p}(\tau),\qquad j\in\mathbb{Z}\quad(2.106)$$

for $j=1$. This allows to check that the relabelling $a''=a'-2p$ just amounts to a shift of $-u$ in the sum over $\theta\in\mathbb{Z}$. Finally, since

$$L^{(-au)}_{au+N}(\tau,\eta)=-L^{(au)}_{N}(\tau,\eta),\tag{2.107}$$

we rewrite

$$L^{(-u[a''+1+p])}_{-2p\theta+p(r'+1)-u(p-1)+u(a''+1+p)}=-L^{(u[a''+1+p])}_{-2p\theta+p(r'+1)-u(p-1)}.\tag{2.108}$$

This enables us to merge the two sums in the square bracket of (2.105) into a sum $\sum_{a''=s+1}^{s+2p}$. The term $a''=p+1$ appears to be missing in (2.105), but this is simply due to the fact that $\Lambda_{p,r'+1,u,p}(\tau)=0$. Our simplest expression for $\Theta^{(1)}$ is therefore,

$$\begin{aligned}\Theta^{(1)}&=\frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac12}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi(s+\frac{p}{u}r)}y^{-\frac{p}{u}}q^{\frac38-\frac{up}{4}}\frac{\vartheta_{1,0}(\mu+\frac12+\frac{\tau}{2},\tau)}{\eta^3(q)}\\&\times\sum_{r'=1}^{u-1}\sum_{a''=s+1}^{s+2p}\sum_{\theta\in\mathbb{Z}}L^{(u[-a''+1+p])}_{2p\theta+p(r'+1)+u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})\Lambda_{a''-1,r'+1,u,p}(\tau)\\&\qquad\times e^{\frac{i\pi p}{u}(2\theta+r')r}y^{a''-\frac{p}{u}(2\theta+r')}q^{-\frac{p}{u}(\frac12+\theta)^2+(\frac12+\theta)(a''-1-\frac{p}{u}r')}\sin\frac{p\pi rr'}{u}.\end{aligned}\tag{2.109}$$

We now proceed and discuss the term $\Psi^{(1)}$ in (2.99). We rearrange the sum on $\theta\in\mathbb{Z}$ as $\sum_{n\in\mathbb{Z}}\sum_{\beta=0}^{u-1}$ where $\theta=un-\beta$ in (2.100) and eliminate the n-dependence of $X^{N=2}_{r',s,u,p;un-\beta-\frac12}(\mu+\frac12,\tau)$ by using the following non-quasi periodicity properties,

$$\begin{aligned}X^{N=2}_{r',s,u,p,;un-\beta-\frac12}(\mu+\frac12,\tau)&=X_{r',s,u,p;-\beta-\frac12}(\mu+\frac12,\tau)-ie^{i\pi s}q^{\frac38-\frac{up}{4}}y^{\frac12}\\&\times\frac{\theta_{1,0}(\mu+\frac12+\frac{\tau}{2},\tau)}{\eta(\tau)^3}\sum_{\ell=0}^{2pn-1}e^{-i\pi\frac{p}{u}(2\beta+r')}y^{s+\ell-\frac{p}{u}(2\beta+r')}q^{-\frac{p}{u}(\frac12+\beta)^2}\\&\qquad\times q^{(\frac12+\beta)(s+\ell-\frac{p}{u}r')}\Lambda_{s+\ell,r'+1,u,p}(\tau)\end{aligned}\tag{2.110}$$

The above formula is valid for $n\geq 1$. For $n\leq -1$, just replace $\sum_{\ell=0}^{2pn-1}$ by $-\sum_{\ell=2pn}^{-1}$. We rewrite $\Psi^{(1)}$ as,

$$\Psi^{(1)}=\Psi^{(2)}+\Theta^{(2)},\tag{2.111}$$

with

$$\Psi^{(2)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi\frac{c}{6}}y^{\frac{c}{6}}$$
$$\times \sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1} e^{-\frac{i\pi p}{u}(2\beta+r'+1)(-r-1)} \sin\frac{p\pi r r'}{u} X^{N=2}_{r',s,u,p,;-\beta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)$$
$$\times \Big[\sum_{n\in\mathbb{Z}}\sum_{s'=1-p}^{p} L^{(u)}_{-2p(un-\beta)+p(r'+1)-u(s'-1)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})\Big]$$
$$= \frac{1}{u}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi\frac{c}{6}}y^{\frac{c}{6}}\sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1} e^{-\frac{i\pi p}{u}(2\beta+r'+1)(-r-1)} \sin\frac{p\pi r r'}{u} X^{N=2}_{r',s,u,p,;-\beta-\frac{1}{2}}(\mu+\frac{1}{2},\tau) \tag{2.112}$$

and

$$\Theta^{(2)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi s}y^{\frac{c}{6}+\frac{1}{2}}q^{\frac{3}{8}-\frac{up}{4}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)} \times \sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1}$$
$$\Big[\sum_{n=1}^{\infty}\sum_{\ell=0}^{2pn-1} - \sum_{n=-\infty}^{-1}\sum_{\ell=2pn}^{-1}\Big]\sum_{s'=1-p}^{p} L^{(u)}_{-2p(un-\beta)+p(r'+1)-u(s'-1)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})$$
$$\times \Lambda_{s+\ell,r'+1,u,p}(\tau)e^{\frac{i\pi p}{u}(2\beta+r'+1)r}y^{s+\ell-\frac{p}{u}(2\beta+r')}q^{-\frac{p}{u}(\frac{1}{2}+\beta)^2}q^{(\frac{1}{2}+\beta)(s+\ell-\frac{p}{u}r')}\sin\frac{p\pi r r'}{u}. \tag{2.113}$$

The above expression for $\Theta^{(2)}$ can be reorganised in such a way it looks similar to $\Theta^{(1)}$. First set $\ell = 2pj + a$ so that $\sum_{\ell=0}^{2pn-1} = \sum_{j=0}^{n-1}\sum_{a=0}^{2p-1}$ (when $n \geq 1$) and $\sum_{\ell=2pn}^{-1} = \sum_{j=n}^{-1}\sum_{a=0}^{2p-1}$ (when $n \leq -1$) . Then relabel $a = a'' - 1 - s$ and use (2.106) together with the result,

$$\sum_{s'=1-p}^{p} L^{(u)}_{-2p(un-\beta)+p(r'+1)-u(s'-1)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})$$
$$= L^{(2pu)}_{-2p(un-\beta)+p(r'+1)+u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p}) \tag{2.114}$$

to write,

$$\Theta^{(2)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi s}y^{\frac{c}{6}+\frac{1}{2}}q^{\frac{3}{8}-\frac{up}{4}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}\sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1}$$
$$[\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}-\sum_{n=-\infty}^{-1}\sum_{j=n}^{-1}]\, L^{(2up)}_{-2p(u[n-j]-(\beta-uj))+p(r'+1)-u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})$$
$$\times\sum_{a''=s+1}^{s+2p}\Lambda_{a''-1,r'+1,u,p}(\tau)e^{\frac{i\pi p}{u}(2[\beta-uj]+r'+1)r}y^{a''-1-\frac{p}{u}(2[\beta-uj]-r')}q^{-\frac{p}{u}(\frac{1}{2}+[\beta-uj])^2}$$
$$\times\, q^{(\frac{1}{2}+[\beta-uj])(a''-1-\frac{p}{u}r')}\sin\frac{p\pi r r'}{u}. \quad (2.115)$$

Replace

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\to\sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty}=\sum_{j=0}^{\infty}\sum_{n'=n-j=1}^{\infty} \quad (2.116)$$

and

$$\sum_{n=-\infty}^{-1}\sum_{j=n}^{-1}\to\sum_{j=-1}^{-\infty}\sum_{n=j}^{-\infty}=\sum_{j=-1}^{-\infty}\sum_{n'=n-j=0}^{-\infty}. \quad (2.117)$$

Also note that,

$$\sum_{n'=1}^{\infty}L^{(2up)}_{-2p(un'-\beta+uj)+p(r'+1)-u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})=$$
$$\int_{-\infty}^{\frac{2\mu-r}{2u}+\frac{s-1}{2p}+\frac{\tau}{2up}(-2p(-\beta+uj)+p(r'+1)+u(1-p))}e^{\frac{2i\pi up\rho^2}{\tau}}d\rho=$$
$$L^{-}_{2p(\beta-uj)+p(r'+1)-u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p}) \quad (2.118)$$

and

$$\sum_{n'=0}^{-\infty}L^{(2up)}_{-2p(un'-\beta+uj)+p(r'+1)-u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})=$$
$$\int^{\infty}_{\frac{2\mu-r}{2u}+\frac{s-1}{2p}+\frac{\tau}{2up}(-2p(-\beta+uj)+p(r'+1)+u(1-p))}e^{\frac{2i\pi up\rho^2}{\tau}}d\rho=$$
$$L^{+}_{2p(\beta-uj)+p(r'+1)-u(1-p)}(\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p}), \quad (2.119)$$

where we have defined

$$L^{\pm}_{n}(\tau,\eta)=\pm i\int_{y+nt}^{\pm\infty}e^{-\pi\frac{x^2}{t}}\Big|_{t=-i\tau,\ y=-i\eta}. \quad (2.120)$$

We have thus arrived at a rewriting of $\Theta^{(2)}$ whose β and j dependence is through the combination $\beta - uj$. Since the domains of summation are $\sum_{j=0}^{\infty}\sum_{\beta=0}^{u-1}$ and

$\sum_{j=-1}^{-\infty}\sum_{\beta=0}^{u-1}$, we can relabel $\beta - uj = \theta$ and consider the sums $\sum_{\theta=-\infty}^{u-1}$ and $\sum_{\theta=u}^{\infty}$ respectively. We therefore write,

$$\Theta^{(2)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi(s+\frac{p}{u}r)}y^{-\frac{p}{u}}q^{\frac{3}{8}-\frac{up}{4}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}$$
$$\times\sum_{r'=1}^{u-1}\Big[\sum_{\theta=-\infty}^{u-1}L^-_{2p\theta+p(r'+1)+u(1-p)}-\sum_{\theta=u}^{\infty}L^+_{2p\theta+p(r'+1)+u(1-p)}\Big](\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})$$
$$\times\sum_{a''=s+1}^{s+2p}\Lambda_{a''-1,r'+1,u,p}(\tau)e^{\frac{i\pi p}{u}(2\theta+r')r}y^{a''-\frac{p}{u}(2\theta+r')}q^{-\frac{p}{u}(\frac{1}{2}+\theta)^2}$$
$$\times q^{(\frac{1}{2}+\theta)(a''-1-\frac{p}{u}r')}\sin\frac{p\pi rr'}{u}. \tag{2.121}$$

This latest rewriting of $\Theta^{(2)}$ allows us to effortlessly add it to $\Theta^{(1)}$ as given in (2.109). We obtain,

$$\Theta^{(1)}+\Theta^{(2)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi(s+\frac{p}{u}r)}y^{-\frac{p}{u}}q^{\frac{3}{8}-\frac{up}{4}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}$$
$$\times\sum_{r'=1}^{u-1}\sum_{a''=s+1}^{s+2p}\Big[\sum_{\theta=-\infty}^{u-1}L^-_{2p\theta+p(r'+1)-u(a''-2)}-\sum_{\theta=u}^{\infty}L^+_{2p\theta+p(r'+1)-u(a''-2)}\Big](\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})$$
$$\times\Lambda_{a''-1,r'+1,u,p}(\tau)e^{\frac{i\pi p}{u}(2\theta+r')r}y^{a''-\frac{p}{u}(2\theta+r')}q^{-\frac{p}{u}(\frac{1}{2}+\theta)^2}$$
$$\times q^{(\frac{1}{2}+\theta)(a''-1-\frac{p}{u}r')}\sin\frac{p\pi rr'}{u}, \tag{2.122}$$

or again, relabelling $a'' = a+1+2p$ and $\theta = \theta' + u$,

$$\Theta^{(1)}+\Theta^{(2)} = \frac{1}{u}\sqrt{\frac{2up}{\tau}}(-i)^{\frac{1}{2}}e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})}e^{i\pi(s+\frac{p}{u}r)}y^{-\frac{p}{u}}q^{\frac{3}{8}-\frac{up}{4}}\frac{\vartheta_{1,0}(\mu+\frac{1}{2}+\frac{\tau}{2},\tau)}{\eta^3(q)}$$
$$\times\sum_{r'=1}^{u-1}\sum_{a=s-2p}^{s-1}\Big[\sum_{\theta'=-\infty}^{-1}L^-_{2p\theta'+p(r'+1)-u(a-1)}-\sum_{\theta'=0}^{\infty}L^+_{2p\theta'+p(r'+1)-u(a-1)}\Big](\frac{\tau}{2up},\frac{2\mu-r}{2u}+\frac{s-1}{2p})$$
$$\times\Lambda_{a,r'+1,u,p}(\tau)e^{\frac{i\pi p}{u}(2\theta'+r')r}y^{a+1-\frac{p}{u}(2\theta'+r')}q^{-\frac{p}{u}(\frac{1}{2}+\theta')^2}$$
$$\times q^{(\frac{1}{2}+\theta')(a-\frac{p}{u}r')}\sin\frac{p\pi rr'}{u}. \tag{2.123}$$

Going back to (2.99) with $\Psi^{(1)}$ given by (2.111), and using (2.112) and (2.123), we can write the S-transform of the non-unitary minimal $N = 2$ characters at central charge $c = 3(1 - \frac{2p}{u})$ as,

$$X^{N=2}_{r,s,u,p}(\frac{\mu}{\tau}, \frac{-1}{\tau}) = \Psi^{(2)} + \Theta^{(1)} + \Theta^{(2)}$$

$$= \frac{1}{u} e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})} e^{i\pi\frac{c}{6}} y^{\frac{c}{6}} \sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1} e^{-\frac{i\pi p}{u}(2\beta+r'+1)(-r-1)} \sin\frac{p\pi r r'}{u} X^{N=2}_{r',s,u,p;-\beta-\frac{1}{2}}(\mu+\frac{1}{2},\tau)$$

$$+ \Theta^{(1)} + \Theta^{(2)} \quad (2.124)$$

It is very interesting to analyse the unitary case from the above formula to get a feeling of how the generalisation to higher values of p complicates matters. **When the parameter p is one**, the $\Theta^{(1)} + \Theta^{(2)}$ contribution vanishes. We are thus left with,

$$X^{N=2}_{r,s,u,1}(\frac{\mu}{\tau}, \frac{-1}{\tau}) =$$

$$\frac{1}{u} e^{i\pi\frac{c}{3}(\frac{\mu^2-\mu}{\tau})} e^{i\pi\frac{c}{6}} y^{\frac{c}{6}} \sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1} e^{-\frac{i\pi}{u}(2\beta+r'+1)(-r-1)} \sin\frac{\pi r r'}{u} X^{N=2}_{r',s,u,1;-\beta-\frac{1}{2}}(\mu+\frac{1}{2},\tau) \quad (2.125)$$

for $1 \leq r \leq u-1$ and $0 \leq s \leq 1$. In view of (2.35), it is sufficient to consider $s = 1$ say. In order to make contact with the formula derived earlier (2.72), write

$$\sum_{r'=1}^{u-1}\sum_{\beta=0}^{u-1} = \sum_{r'=1}^{u-1}(\sum_{\beta=0}^{r'-1} + \sum_{\beta=r'}^{u-1}) \quad (2.126)$$

in (2.125) and relabel $r' = u - r'', \beta = \beta' - r'' + u$ in the second term. After using the properties (2.39) and (2.40), we exactly obtain the result (2.72).

2.5 Summary

In this chapter we looked at some basic properties of the $N = 2$ superconformal algebra. We showed that when the spectral flow is applied to unitary characters, it generates a finite number of characters.

We also discussed admissible (non - unitary) characters which were not quasi - periodic under the spectral flow. The $N = 2$ characters can not be expressed in tems of θ - functions.

We have found expressions for the modular S transformation of $N = 2$ character at $c = 3(1 - \frac{2p}{u})$. This has allowed us to calculate all modular transformations for the cases $p = 1$ and $p \neq 1$.

Chapter 3

The affine superalgebra $\widehat{sl}(2 \mid 1; \mathbb{C})$

.

3.1 Introduction

As mentioned in the introduction, the affine superalgebra $\widehat{sl}(2 \mid 1; \mathbb{C})$ is relevant in a variety of physical contexts, and the ultimate purpose of this chapter is to present character formulas which encode the content of a particular class of highest weight irreducible representations which emerge when the level of $\widehat{sl}(2 \mid 1; \mathbb{C})$ is of the form $k = \frac{p}{u} - 1$ with p, u two coprime positive integers. The irreducible highest weight representations of affine (super)algebras at level $k = \frac{p}{u} - h^\vee$ where $h^\vee$ is the dual Coxeter number are called *admissible* [64] and [65]. We will not derive the character formulas here, but introduce the necessary background to provide a good feeling of their mathematical meaning.

In section 2, we give a brief description of the root and weight lattices of the $\widehat{sl}(2 \mid 1; \mathbb{C})$ algebra and identify a variety of automorphisms of this algebra. These are relevant when studying the structure of modules, as in general, applying algebra automorphisms to modules gives non-isomorphic modules. Furthermore, the automorphisms play an important role in the analysis of Verma module singular vectors and their embedding structure. They are therefore relevant in the construction of character formulas.

Most of this chapter is based on published works, except for the character formulas in section 3, which are written here for the first time for admissible representations at level $k = \frac{p}{u} - 1$ with p different from one.

3.2 The affine superalgebra $\widehat{sl}(2 \mid 1; \mathbb{C})$

The affine superalgebra $\widehat{sl}(2 \mid 1; \mathbb{C})$ is generated by eight currents of conformal dimension one. The currents $J^{\pm}(z), J^3(z)$ and $U(z)$ are bosonic and generate the even affine subalgebra $\widehat{sl(2)} \times \widehat{u(1)}$ while the remaining currents are fermionic and are labelled $j^{\pm}(z), j^{\pm'}(z)$. If we assume the eight currents satisfy periodic boundary conditions of the type,

$$\mathcal{J}(e^{2i\pi} z) = \mathcal{J}(z) \tag{3.1}$$

so that their Laurent expansion is given by,

$$\mathcal{J}(z) = \sum_{n \in \mathbb{Z}} \mathcal{J}_n z^{-n-1}, \tag{3.2}$$

their non vanishing (anti)commutation relations are,

$$\begin{array}{rclrcl}
[J^+_m, J^-_n] &=& 2J^3_{m+n} + \tilde{k} m \delta_{m+n,0}, & [J^3_m, J^{\pm}_n] &=& \pm J^{\pm}_{m+n}, \\
\left[J^{\pm}_m, j'^{\mp}_n\right] &=& \pm j^{\pm}_{m+n}, & [J^{\pm}_m, j^{\mp}_n] &=& \mp j'^{\pm}_{m+n}, \\
\left[2J^3_m, j'^{\pm}_n\right] &=& \pm j'^{\pm}_{m+n}, & [2J^3_m, j^{\pm}_n] &=& \pm j^{\pm}_{m+n}, \\
\left[2U_m, j'^{\pm}_n\right] &=& \pm j'^{\pm}_{m+n}, & [2U_m, j^{\pm}_n] &=& \mp j^{\pm}_{m+n}, \\
[J^3_m, J^3_n] &=& \frac{\tilde{k}}{2} m \delta_{m+n,0}, & [U_m, U_n] &=& -\frac{\tilde{k}}{2} m \delta_{m+n,0}, \\
\left[j'^+_m, j'^-_n\right]_+ &=& U_{m+n} - J^3_{m+n} - m\tilde{k}\delta_{m+n,0} & & & , \\
[j^+_m, j^-_n]_+ &=& U_{m+n} + J^3_{m+n} + m\tilde{k}\delta_{m+n,0} & & & \\
\left[j'^+_m, j^+_n\right]_+ &=& J^+_{m+n}, & \left[j'^-_m, j^-_n\right]_+ &=& J^-_{m+n}.
\end{array} \tag{3.3}$$

The zero-mode generators close among themselves to form the basic, classical simple complex superalgebra $sl(2/1)$ in the classification of Kac. Its Cartan subalgebra is generated by J^3_0 and U_0, and the two-dimensional root diagram may be represented in Minkowski space with the fermionic roots along the light-cone directions, as illustrated in Fig.3.1.

The step operators corresponding to the roots α_1, α_2 and $(\alpha_1 + \alpha_2)$ are $j^{+'}, j^+$ and J^+ respectively. A particularity of superalgebras is that there exist several

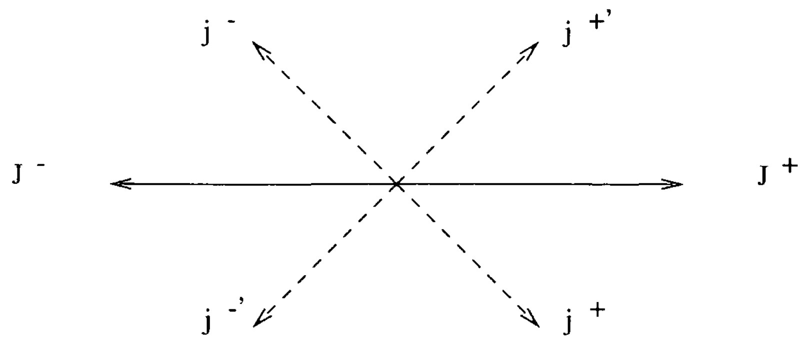

Figure 3.1: The $s\ell(2|1)$ root diagram.

choices of simple roots which are not related by a Weyl transformation. The simple roots are a useful subset of roots because all positive (resp. negative) roots can be written as linear combinations of these simple roots with positive (resp. negative) coefficients. Each independent set of simple roots contains as many roots as the rank of the algebra (in our case, the rank is two since the maximal number of commuting generators is two). In [37], the simple roots were taken to be the fermionic roots α_1 and α_2, with the scalar products,

$$(\alpha_1, \alpha_2) = 1, \qquad \alpha_1^2 = \alpha_2^2 = 0. \tag{3.4}$$

These roots are therefore isotropic (zero norm) and the notion of fundamental weight as introduced in the context of affine Lie algebras does not have a straightforward generalisation. We will come back later in this section on a description of the affine root and weight systems, as these are important when discussing highest weight states and singular vectors within Verma modules, and therefore when describing the characters of irreducible representations of the affine Lie superalgebra.

Here we choose as simple roots the bosonic root $\tilde{\alpha_1}$ and the fermionic root $\tilde{\alpha_2} - \tilde{\alpha_1}$. They are related to the previous set by,

$$\tilde{\alpha_1} = \alpha_1 + \alpha_2, \qquad \tilde{\alpha_2} - \tilde{\alpha_1} = -2\alpha_2. \tag{3.5}$$

The eigenvalue k of the central element $\tilde{k}$ which appears in the commutation relations (3.3) is the *level* of the algebra. A priori, it can be any complex number, but we will mainly discuss here a particular class of levels of the form,

$$k = \frac{p}{u} - 1, \qquad \text{with}\, p, u \,\text{coprime and}\; p, u \in \mathbb{N}. \tag{3.6}$$

Note that the Sugawara energy-momentum tensor is given by,

$$T_{Sug}(z) = \frac{1}{k+1}\left[J^3J^3 - UU + J^+J^- + j^{+'}j^{-'} - j^+j^-\right](z), \tag{3.7}$$

and its Laurent modes generate a Virasoro algebra with zero central charge. This very particular value of central charge is related to the fact that $\widehat{sl}(2 \mid 1;\mathbb{C})$ has an equal number of even and odd generators. Indeed, in the context of affine superalgebras, the central charge associated to the Sugawara tensor is given by the formula,

$$c = \frac{k\ sdim\ \mathcal{G}}{k+h^\vee}, \tag{3.8}$$

with *sdim* giving the superdimension of the algebra $\mathcal{G}$, that is the difference between the numbers of even (bosonic) and odd (fermionic) generators. It is quite remarkable that the central charge vanishes irrespectively of the value of the level.

The affine $\widehat{sl}(2 \mid 1;\mathbb{C})$ algebra possesses several automorphisms. We have not studied them all in details, but we would like to list those we are aware of. One of them is the spectral flow and will become relevant when we discuss the modular transformations of $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters in Chapter 4. The first automorphism is given by,

$$\alpha: \quad \begin{array}{lll} j^{+'}_m \mapsto j^-_m, & j^+_m \mapsto j^{-'}_m, & J^+_m \mapsto J^-_m, \\ j^-_m \mapsto j^{+'}_m, & j^{-'}_m \mapsto j^+_m, & J^-_m \mapsto J^+_m, \\ U_n \mapsto U_n & J^3_n \mapsto -J^3_n. & \end{array} \tag{3.9}$$

The β automorphism is,

$$\beta: \quad \begin{array}{lll} j^+_m \mapsto j^{+'}_m, & j^{-'}_m \mapsto -j^-_m, & J^+_m \mapsto J^+_m, \\ j^{+'}_m \mapsto j^+_m, & j^{-'}_- \mapsto -j^-_m. & \\ U_n \mapsto -U_n & J^3_n \mapsto J^3_n. & \end{array} \tag{3.10}$$

The transformation,

$$\mathcal{U}_\theta: \quad \begin{array}{lll} j^{+'}_m \mapsto j^{+'}_{m-\theta}, & j^+_m \mapsto j^{+'}_{m+\theta}, & U_m \mapsto U_m - k\theta\delta_{m,0}, \\ j^{-'}_m \mapsto j^{-'}_{m+\theta}, & j^-_m \mapsto j^-_{m-\theta} & \end{array} \tag{3.11}$$

is an automorphism of $\widehat{sl}(2 \mid 1;\mathbb{C})$ whenever $\theta \in \mathbb{Z}$. If $\theta \in \mathbb{Z}+\frac{1}{2}$, the transformation $\mathcal{U}_\theta$ is a mapping into an isomorphic algebra. We call $\mathcal{U}_\theta$ a spectral flow . Note that $\mathcal{U}_{\pm\frac{1}{2}}$ maps integer moded fermionic (odd) generators into half-integer moded ones, and we continue to refer to the sector of the algebra with half-integer moded fermionic generators as being the 'Neveu-Schwarz' sector. However, it should be noted that because all currents in $\widehat{sl}(2 \mid 1;\mathbb{C})$ have conformal dimension one, the Neveu-Schwarz currents have a branch cut while the Ramond fields are single valued, in contrast with the situation described in the previous chapter for $N=2$.

Another type of automorphisms acts as,

$$\begin{array}{llll} A_\eta : j^{+'}_m \mapsto j^{+'}_{m+\eta}, & j^{+}_m \mapsto j^{+}_{m+\eta}, & J^{+}_m \mapsto J^{+}_{m+2\eta}, & J^{3}_m \mapsto J^{3}_m - k\eta\delta_{m,0} \\ \quad\; j^{-'}_m \mapsto j^{-'}_{m-\eta}, & j^{-}_m \mapsto j^{-}_{m-\eta}, & J^{-}_m \to J^{-}_{m-2\eta}, & U_m \mapsto U_m \end{array} \tag{3.12}$$

The composition $U_{\frac{1}{2}} \circ A_{\frac{1}{2}}$ is an automorphism, and we also have the following properties,

$$\begin{aligned} \alpha^2 = 1, \qquad \beta^2 = 1, \qquad (\alpha\beta)^4 = 1, \\ \alpha\mathcal{U}_\theta = \mathcal{U}_\theta\alpha \qquad (\beta\mathcal{U}_\theta)^2 = 1. \end{aligned} \tag{3.13}$$

We end up this section by introducing the quantum numbers associated with a generic state $|\lambda\rangle$ of a $\widehat{sl}(2 \mid 1;\mathbb{C})$ module, and by presenting the condition for a singular vector to exist in a Verma module with highest weight state $|\Lambda\rangle$.

Given an affine Lie superalgebra, it is always possible to construct a Sugawara energy- momentum tensor whose modes obey the commutation relations of a Virasoro algebra. The resulting algebraic structure is a semi-direct product of the affine superalgebra and the Virasoro algebra, and the associated Cartan subalgebra is spanned by the Cartan algebra of the affine superalgebra and a derivative operator d associated to the zero mode of the Sugawara energy- momentum tensor. The commutation relations of this operator d with all the generators X_n of $\widehat{sl}(2 \mid 1;\mathbb{C})$ (except the central element $\tilde{k}$) are of the form,

$$[d, X_n] = nX_n. \tag{3.14}$$

In the case of $\widehat{sl}(2 \mid 1;\mathbb{C})$, the Cartan algebra is spanned by the set $\{J_0^3, U_0, \tilde{k}, d\}$ where d is the zero mode of (3.7). We introduce $\tilde{\alpha_1}, \tilde{\alpha_2}, \lambda_0, \delta$ as the dual to these Cartan elements in accordance to,

$$\langle \tilde{\alpha_1}, J_0^3 \rangle = 1, \qquad \langle \tilde{\alpha_2}, U_0 \rangle = 1, \qquad \langle \lambda_0, \tilde{k} \rangle = 1, \qquad \langle \delta, d \rangle = 1. \tag{3.15}$$

Their scalar products are defined to be,

$$\begin{aligned}
&(\lambda_0,\lambda_0) = 0, \qquad (\lambda_0,\tilde{\alpha_1}) = 0, \qquad (\lambda_0,\tilde{\alpha_2}) = 0, \qquad (\lambda_0,\delta) = 1,\\
&(\tilde{\alpha_1},\tilde{\alpha_1}) = 2, \qquad (\tilde{\alpha_1},\tilde{\alpha_2}) = 0, \qquad (\tilde{\alpha_1},\delta) = 0,\\
&(\tilde{\alpha_2},\tilde{\alpha_2}) = -2, \qquad (\tilde{\alpha_2},\delta) = 0, \qquad (\delta,\delta) = 0.
\end{aligned} \tag{3.16}$$

We choose to express a generic weight λ as a linear combination of these dual elements,

$$\lambda = h_- \tilde{\alpha_1} + h_+ \tilde{\alpha_2} + k\lambda_0 + \Delta(h_-, h_+, k)\delta, \tag{3.17}$$

and interpret the coefficients as the isospin (h_-), hypercharge (h_+)and conformal weight (Δ) of the weight λ. k is the level at which we consider $\widehat{sl}(2 \mid 1;\mathbb{C})$.

Note that the weight lattice P of the affine superalgebra $\widehat{sl}(2 \mid 1;\mathbb{C})$ may therefore be described as,

$$P = \mathbb{Z}\lambda_0 \oplus \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \oplus \mathbb{Z}\delta, \tag{3.18}$$

where

$$\lambda_1 = \lambda_0 + \frac{\tilde{\alpha_1}}{2}, \qquad \lambda_2 = \lambda_0 + \frac{\tilde{\alpha_2}}{2}. \tag{3.19}$$

The simple roots are taken to be,

$$\tilde{\alpha_1}, \qquad \tilde{\alpha_2} - \tilde{\alpha_1}, \qquad \text{and} \qquad \tilde{\alpha_0} = \tilde{\alpha_2} - \tilde{\alpha_1} + \delta. \tag{3.20}$$

The real non-isotropic positive roots are

$$\tilde{\alpha_1} + (s-1)\delta, \qquad -\tilde{\alpha_1} + s\delta, \tag{3.21}$$

while the isotropic ones are,

$$\begin{aligned}
&\tilde{\alpha_2} - \tilde{\alpha_1} + (s-1)\delta, \qquad \tilde{\alpha_1} - \tilde{\alpha_2} + s\delta,\\
&\tilde{\alpha_1} + \tilde{\alpha_2} + (s-1)\delta, \qquad -\tilde{\alpha_1} - \tilde{\alpha_2} + s\delta,
\end{aligned} \tag{3.22}$$

where $s \in \mathbb{N}$. We also define the affine Weyl vector ρ as follows,

$$\rho = \lambda_0 + \lambda_1 - \lambda_2 \tag{3.23}$$

and therefore,

$$(\rho, \delta) = 1, \qquad (\rho, \tilde{\alpha}_i) = 1, \quad i = 0, 1, 2. \tag{3.24}$$

This enables us to introduce R_α, which is a shifted reflection with respect to a real root α. If Λ is a highest weight, the condition for a singular vector to exist in the Verma module is,

$$R_\alpha \Lambda \equiv \Lambda - \frac{2(\Lambda + \rho, \alpha)}{(\alpha, \alpha)} \alpha = \Lambda - r\alpha, \tag{3.25}$$

where $r \in \mathbb{N}$. This condition is equivalent to

$$(\Lambda + \rho, \alpha) = r \in \mathbb{N} \tag{3.26}$$

for the non-isotropic roots (3.21). Therefore, for this subset of positive real roots, one finds, using the notations (3.17), that the isospin h_- of the highest weight state must take one of the following values in order for a singular vector to exist, i.e. in order to get a zero of the Kac-Wakimoto determinant,

$$h_- = \frac{r-1}{2} - \frac{s-1}{2}(k+1) \qquad \text{for} \qquad \tilde{\alpha}_1 + (s-1)\delta, \qquad r, s \in \mathbb{N}, \tag{3.27}$$

or

$$h_- = -\frac{r+1}{2} + \frac{s}{2}(k+1) \qquad \text{for} \qquad -\tilde{\alpha}_1 + s\delta, \qquad r, s \in \mathbb{N}. \tag{3.28}$$

For the isotropic roots (3.22) the reflection R_α does not exist, but one can use the condition

$$(\Lambda + \rho, \alpha) = 0 \tag{3.29}$$

instead. This produces relations between the isospin and the hypercharge of the highest weight Λ, which must be satisfied if a singular vector has to exist,

$$h_- + h_+ = \frac{1}{2}(s-1)(k+1), \qquad h_- + h_+ = -\frac{1}{2}s(k+1) \tag{3.30}$$

or

$$h_- - h_+ = -\frac{1}{2}(s-1)(k+1), \qquad h_- - h_+ = \frac{1}{2}s(k+1). \tag{3.31}$$

In conclusion, a $\widehat{sl}(2 \mid 1; \mathbb{C})$ highest weight at level k is characterised by its isospin h_-, hypercharge h_+ and conformal weight Δ. We write $|\Lambda\rangle = |h_-, h_+, k\rangle$ with

$$U_0|h_-, h_+, k\rangle = h_+|h_-, h_+, k\rangle, \qquad J_0^3|h_-, h_+, k\rangle = h_-|h_-, h_+, k\rangle \tag{3.32}$$

and

$$L_0|h_-, h_+, k\rangle = \Delta_{h_-,h_+,k}|h_-, h_+, k\rangle, \tag{3.33}$$

where the conformal weight $\Delta_{h_-,h_+,k}$ can be read off the expression (3.7) for the Sugawara energy-momentum tensor. It is given by,

$$\Delta_{h_-,h_+,k} = \frac{h_-^2 - h_+^2}{k+1}. \tag{3.34}$$

The annihilation conditions for the highest weight state are, [66]

$$j'^{+}_{0}|h_-, h_+, k\rangle = 0, \qquad j_0^+|h_-, h_+, k\rangle = 0, \qquad J_1^-|h_-, h_+, k\rangle = 0. \tag{3.35}$$

3.3 Twisted highest weight states and character formulas

The existence of an automorphism group leads to a freedom in choosing the type of annihilation conditions imposed on highest weight states of Verma modules.

A twisted Verma module $V_{h_-,h_+,k;\theta}$ (with integer twist θ) over the level $k = \frac{p}{u} - 1$ $\widehat{sl}(2 \mid 1; \mathbb{C})$ algebra is freely generated by $j^{+'}_{\leqslant -\theta-1}$, $j^{+}_{\leqslant \theta-1}$, $j^{-'}_{\leqslant \theta}$, $j^{-}_{\leqslant -\theta}$, $J^{+}_{\leqslant -1}$, $J^{-}_{\leqslant 0}$, $U_{\leqslant -1}$, and $J^3_{\leqslant -1}$ from the twisted highest weight state $|h_-, h_+, k; \theta>$ satisfying the annihilation conditions

$$j^{+\prime}_{-\theta}|h_-, h_+, k; \theta >= 0, \qquad j_\theta^+|h_-, h_+, k; \theta >= 0, \qquad J_1^-|h_-, h_+, k; \theta >= 0. \tag{3.36}$$

The particular case when $\theta = 0$ corresponds to the *untwisted* highest weight state conditions discussed in the previous section (3.35). The twisted highest weight conditions (3.36) are mapped into one another by the spectral flow (3.11). The action of the latter on $\{J^3_m, U_m, L_m\}$ is given by,

$$\begin{aligned} \mathcal{U}_\theta : J^3_m &\mapsto J^{3'}_m = J^3_m, \qquad U_m \mapsto U'_m = U_m - k\theta\delta_{m,0}, \\ L_m &\mapsto L'_m = L_m + 2\theta U_m - k\theta^2\delta_{m,0}, \end{aligned} \tag{3.37}$$

and therefore, in view of (3.32) one has,

$$J_0^{3'}|h_-,h_+,k;\theta\rangle = h_-|h_-,h_+,k;\theta\rangle, \qquad U_0'|h_-,h_+,k;\theta\rangle = (h_+ - k\theta)|h_-,h_+,k;\theta\rangle \tag{3.38}$$

and

$$L_0'|h_-,h_+,k;\theta\rangle = \Delta_{h_-,h_+,k,\theta}|h_-,h_+,k;\theta\rangle \tag{3.39}$$

where the conformal weight of this twisted highest weight state is,

$$\Delta_{h_-,h_+,k,\theta} = \frac{h_-^2 - h_+^2}{k+1} + 2\theta h_+ - k\theta^2. \tag{3.40}$$

Note the following spectral flow action on the highest weight state and the Verma module,

$$\begin{aligned} \mathcal{U}_{\theta'}|h_-,h_+,k;\theta\rangle &= |h_-,h_+,k;\theta+\theta'\rangle, \\ \mathcal{U}_{\theta'}V_{h_-,h_+,k;\theta} &= V_{h_-,h_+,k;\theta+\theta'}. \end{aligned} \tag{3.41}$$

The character of an untwisted Verma module of highest weight $|h_-,h_+,k\rangle$ is formally defined as a trace over the module $V_{h_-,h_+,k}$,

$$\chi^V_{h_-,h_+,k}(z,\zeta,q) = Tr_{V_{h_-,h_+,k}}(q^{L_0} z^{J_0^3} \zeta^{U_0}), \tag{3.42}$$

where q, z and ζ are three complex variables, $q = e^{2i\pi\tau}$, $z = e^{2i\pi\nu}$ and $\zeta = e^{2i\pi\rho}$, with $\tau, \nu, \rho \in \mathbb{C}$ and $Im(\tau) > 0$ for convergence purposes. Note that the trace apparently does not include a Casimir factor $q^{-\frac{c}{24}}$, but the Virasoro central charge c for $\widehat{sl}(2 \mid 1;\mathbb{C})$ is actually zero, as argued previously (3.8). The twisted characters are obtained from the untwisted ones by spectral flow. Since

$$Tr_V(q^{L_0'} z^{J_0^{3'}} \zeta^{U_0'}) = Tr_V(q^{L_0+2\theta U_0 - k\theta^2} z^{J_0^3} \zeta^{U_0 - k\theta}) = \zeta^{-k\theta} q^{-k\theta^2} Tr_V(q^{L_0} z^{J_0^3} (\zeta q^{2\theta})^{U_0}), \tag{3.43}$$

the character of a twisted Verma module is given by,

$$\chi^V_{h_-,h_+,k;\theta}(z,\zeta,q) = \zeta^{-k\theta} q^{-k\theta^2} \chi^V_{h_-,h_+,k}(z,\zeta q^{2\theta},q), \tag{3.44}$$

where the untwisted Verma module character is given by [27],

$$\chi^V_{h_-,h_+,k}(z,\zeta,q) = z^{h_-}\zeta^{h_+} q^{\frac{h_-^2-h_+^2}{k+1}} \frac{\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta(q)^3}, \tag{3.45}$$

where the theta functions have been introduced earlier (2.25).

One of the themes of this thesis is the study of the modular properties of characters corresponding to a particular class of irreducible representations of $\widehat{sl}(2 \mid 1;\mathbb{C})$ at level $k = \frac{p}{u} - 1$, with p, u coprime. These irreducible characters may be found by constructing resolutions, but this in turn requires analyzing Verma module singular vectors and embeddings, taking advantage of the $\widehat{sl}(2 \mid 1;\mathbb{C})$ automorphisms we discovered earlier. It is beyond the scope of this thesis to explain in details the steps which lead from the character of twisted highest weight state Verma modules of the type (3.44) to character formulas for the irreducible, admissible representations of interest to us. These steps are presented in great details in [66], and we now provide the character formulas which can be derived from that publication. They are the most important formulas of the present chapter and are the starting point of the next chapter, when we discuss their modular properties. They were provided to us by A. Semikhatov and A. Taormina and will be the object of intense study in a forthcoming publication based on the results in this thesis [40].

For specific values of the quantum numbers h_- and h_+ (3.27, 3.28,3.30,3.31), the untwisted Verma module $V_{h_-,h_+,k}$ contains singular vectors which are responsible for the reducibility of the Verma module. Indeed, each singular vector may be viewed as the highest weight state of a module of zero norm states which should be eliminated from the Verma module in order to obtain an irreducible representation. We consider the $\widehat{sl}(2 \mid 1;\mathbb{C})$ algebra at level $k = \frac{p}{u} - 1$ and the class of representations with

$$h_- = \frac{r-1}{2} - \frac{s-1}{2}\frac{p}{u}, \qquad 1-p \le r \le p, \qquad 1 \le s \le u, \tag{3.46}$$

and

$$h_- - h_+ = \frac{p}{u}. \tag{3.47}$$

The character formula (3.45) acquires a 'corrective' factor $\psi_{r,s,u,p}(z,\zeta,q)$ which takes into account the modding out by submodules with a singular vector as highest weight state. Explicitly, the untwisted $\widehat{sl}(2 \mid 1;\mathbb{C})$ admissible characters are given by,

$$\begin{aligned} \chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p}(z,\zeta,q) &= z^{\frac{r-1}{2}-\frac{s-1}{2}\frac{p}{u}} \zeta^{\frac{r-1}{2}-\frac{s+1}{2}\frac{p}{u}} q^{r-1-s\frac{p}{u}} \\ &\times \frac{\theta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}},q)\theta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}},q)}{\theta_{1,1}(z,q)\eta(q)^3}\psi_{r,s,u,p}(z,\zeta,q), \end{aligned} \tag{3.48}$$

with

$$\psi_{r,s,u,p}(z,\zeta,q) = \sum_{m\in\mathbb{Z}} q^{m^2up-mu(r-1)}\left(\frac{q^{mp(s-1)}z^{-mp}}{1+z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{mu-1}} - q^{(s-1)(r-1)}z^{1-r}\frac{q^{-mp(s-1)}z^{mp}}{1+z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{mu-s}}\right). \quad (3.49)$$

The integer-twisted characters are labelled by integers r, s, θ such that

$$1-p \leq r \leq p, \qquad 1 \leq s \leq u, \qquad \theta \in \mathbb{Z}. \quad (3.50)$$

They are given by,

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta}(z,\zeta,q) = \zeta^{-k\theta}q^{-k\theta^2}X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p}(z,\zeta q^{2\theta},q), \quad (3.51)$$

i.e.

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta}(z,\zeta,q) = z^{\frac{r-1}{2}-\frac{s-1}{2}\frac{p}{u}}\zeta^{\frac{r-1}{2}-\frac{s+1}{2}\frac{p}{u}-\theta\frac{p}{u}}q^{(\theta+1)(r-1-\frac{p}{u}(s+\theta))} \times \frac{\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta^3(q)}\psi_{r,s,u,p}(z,\zeta q^{2\theta},q). \quad (3.52)$$

The fundamental range (3.50) is consistent with the following properties enjoyed by the characters. First of all, when $n \in \mathbb{Z}$,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{2np+r,s,u,p;\theta}(z,\zeta,q) = \chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta-nu}(z,\zeta,q) = \chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s-2nu,u,p;\theta}(z,\zeta,q), \quad (3.53)$$

and

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{-(2n+1)p+r,s,u,p;\theta}(z,\zeta,q) = \chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s+u,u,p;\theta+nu}(z,\zeta,q) = \chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s+(2n+1)u,u,p;\theta}(z,\zeta,q) = \chi^{\widehat{sl}(2|1;\mathbb{C})}_{r-p,s,u,p;\theta+nu}(z,\zeta,q). \quad (3.54)$$

Second of all, exactly as in the case of $N=2$, the function $\psi_{r,s,u,p;\theta}(z,\zeta,q) \equiv \psi_{r,s,u,p}(z,\zeta q^{2\theta},q)$ is not periodic in the spectral flow parameter θ for p other than one, and this behaviour is at the root of complications in the derivation of the modular transformations of the $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters. For instance, in the case where the twist θ is zero, one has, for $n \leq -1$,

$$\psi_{r,s,u,p}(z,\zeta q^{2nu},q) = \zeta^{np}q^{upn^2+n[p(s+1)-u(r-1)]}\psi_{r,s,u,p}(z,\zeta,q) + \sum_{\ell=0}^{-2pn-1}(-1)^{\ell}\zeta^{-\frac{1}{2}\ell}z^{-\frac{1}{2}\ell}q^{-\frac{up}{4}-\ell(upn+1)}\,\Lambda_{r-1-\ell+p,s+u,u,p}(z^{-1},q) \quad (3.55)$$

where we have defined

$$\Lambda_{r,s,u,p}(z,q) = \vartheta_{1,0}(z^p q^{-p(s-1)+ru}, q^{2up}) - q^{(s-1)r} z^{-r} \vartheta_{1,0}(z^p q^{-p(s-1)-ru}, q^{2up}). \quad (3.56)$$

We see that the second term spoils the quasi-periodicity of the function $\psi_{r,s,u,p}(z,\zeta,q)$ under the shift $\zeta \to \zeta q^{2nu}$ when p is different from one. It vanishes when $p = 1$, restoring the quasi-periodicity in that case, and rendering the discussion of modular properties much simpler.

It is worthwhile remarking that the beta automorphism (3.10) acts on any module M as,

$$X^{\beta M}_{r,s,u,p}(z,\zeta,q) = X^{M}_{r,s,u,p}(z,\zeta^{-1},q), \quad (3.57)$$

and therefore, we have

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta}(z,\zeta^{-1},q) = -X^{\widehat{sl}(2|1;\mathbb{C})}_{1-r,s,u,p;\theta-s-1}(z,\zeta,q). \quad (3.58)$$

We have mainly discussed so far one sector of the theory (we shall call it the 'Ramond' sector by tradition), but modular transformations mix the Ramond sector with other sectors which we now discuss briefly.

Instead of considering integer twists θ, let us consider twists of the form $\theta \pm \frac{1}{2}, \theta \in \mathbb{Z}$, and write, in accordance with (3.51),

$$\begin{aligned} X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta\pm\frac{1}{2}}(z,\zeta,q) &= \zeta^{-k(\theta\pm\frac{1}{2})} q^{-k(\theta\pm\frac{1}{2})^2} X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p}(z,\zeta q^{2\theta\pm 1},q) \\ &= \zeta^{\mp\frac{k}{2}} q^{-\frac{k}{4}} X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta}(z,\zeta q^{\pm 1},q). \end{aligned} \quad (3.59)$$

The above characters belong to a new sector called the 'Neveu-Schwarz' sector. Two more sectors are relevant in the discussion of modular properties. The corresponding characters are usually called supercharacters and are given by,

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta}(z,-\zeta,q) \qquad \text{and} \qquad X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;\theta\pm\frac{1}{2}}(z,-\zeta,q). \quad (3.60)$$

The formula (3.52) for an infinite family of twisted admissible $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters is the most important expression in this chapter. It generalises character formulas which were derived previously when the level of the affine Lie superalgebra was of the form $k = \frac{1}{u} - 1$ (the case we refer to as the $p = 1$ case).

It is instructive to set $p = 1$ in (3.52) and see how the characters compare with those available in the literature [66] [29].

When $p = 1$, the label r takes the values $r = 0$ and $r = 1$ according to the range (3.50). Let us first consider the characters (3.52) for $r = 1$. It was shown in [66] that these characters can be written in an infinite product form (using a residue analysis for the single poles in $\psi_{1,s,u,1}(z,\zeta,q)$), namely,

$$\begin{aligned} X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta,q) &= z^{-\frac{s-1}{2u}}\zeta^{-\frac{s+1}{2u}-\frac{\theta}{u}}q^{-\frac{1}{u}(\theta+1)(\theta+s)}F(z,\zeta,q) \\ &\quad\times\frac{\eta(q^u)^3\vartheta_{1,1}(zq^{-s+1},q^u)}{\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{\theta+1},q^u)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{-\theta-s},q^u)}, \end{aligned} \tag{3.61}$$

with

$$F(z,\zeta,q) = \frac{\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta(q)^3}. \tag{3.62}$$

The above expression is extremely convenient to prove the periodicity of admissible $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters when the twist θ is shifted by an integer amount of u. One has indeed (see appendix A),

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta+nu}(z,\zeta,q) = \zeta^{-knu}q^{-kn^2u^2}X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta q^{2nu},q) = X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta,q), \tag{3.63}$$

since

$$\begin{aligned} \vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{\theta+1+nu},q^u) &= z^{-\frac{n}{2}}\zeta^{-\frac{n}{2}}q^{-n(\theta+1+\frac{u}{2}(n+1))}\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{\theta+1},q^u), \\ \vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{-\theta-s-nu},q^u) &= z^{\frac{n}{2}}\zeta^{-\frac{n}{2}}q^{-n(\theta+s+\frac{u}{2}(n-1))}\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{-\theta-s},q^u), \end{aligned} \tag{3.64}$$

and

$$F(z,\zeta q^{2nu},q) = \zeta^{-nu}q^{-n^2u^2}F(z,\zeta,q). \tag{3.65}$$

The product formula (3.61) is also useful to relate the characters labelled by $r = 1$ to the characters labelled by $r = 0$. The first equality in (3.54) for $n = 0$ and $p = 1, r = 1$ gives,

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{0,s,u,1;\theta}(z,\zeta,q) = X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s+u,u,1;\theta}(z,\zeta,q). \tag{3.66}$$

But

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s+u,u,1;\theta}(z,\zeta,q) = -X^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta,q), \tag{3.67}$$

as can be easily checked using (3.61). This relation is a manifestation of the fact that the beta automorphism (3.10) acts trivially when $p = 1$. So we conclude that *when p is one*, all independent characters are labelled by $r = 1$ and,

$$1 \leq s \leq u, \qquad \text{and} \qquad 0 \leq \theta \leq u-1. \tag{3.68}$$

This gives a *finite* number of irreducible characters, namely u^2 characters, which are in one-to-one correspondence with those introduced in [66] [67]. There, $\frac{1}{2}u(u+1)$ of them formed the class IV and $\frac{1}{2}u(u-1)$ of them formed the class V. One has,

$$\begin{aligned} \mathcal{X}^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta,q) &= \chi^{R,IV}_{s-1,u-\theta-1}(z,\zeta,q) \qquad \text{for } 1 \leq s \leq u,\ u-s \leq \theta \leq u-1 \\ &= \chi^{R,V}_{u-s-1-\theta,\theta}(z,\zeta,q) \qquad \text{for } 1 \leq s \leq u-1,\ 0 \leq \theta \leq u-s-1. \end{aligned}$$

The most direct check of the above correspondence is to rewrite (3.61) as two equivalent expressions, using properties of theta functions of the type described in (3.64). The first one is,

$$\begin{aligned} \mathcal{X}^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta,q) &= z^{-\frac{s-1}{2u}}\zeta^{-\frac{s+1}{2u}-\frac{\theta}{u}+1}q^{-\frac{1}{u}(\theta+1)(\theta+s)+2\theta+s+1-u}F(z,\zeta,q) \\ &\times \frac{\eta(q^u)^3\vartheta_{1,1}(zq^{-s+1},q^u)}{\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{\theta+1-u},q^u)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{u-\theta-s},q^u)}, \end{aligned} \tag{3.69}$$

which is precisely the product formula obtained in ([29,66]) for class IV characters. The second rewriting of (3.61) is,

$$\begin{aligned} \mathcal{X}^{\widehat{sl}(2|1;\mathbb{C})}_{1,s,u,1;\theta}(z,\zeta,q) &= -z^{-\frac{s-1}{2u}+\frac{1}{2}}\zeta^{-\frac{s+1}{2u}-\frac{\theta}{u}+\frac{1}{2}}q^{-\frac{1}{u}(\theta+1)(\theta+s)+\theta+1}F(z,\zeta,q) \\ &\times \frac{\eta(q^u)^3\vartheta_{1,1}(zq^{u-s+1},q^u)}{\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{\theta+1},q^u)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{u-\theta-s},q^u)}, \end{aligned} \tag{3.70}$$

which is the product formula for class V.

So we have presented new formulas for admissible characters of the affine Lie superalgebra $\widehat{sl}(2 \mid 1;\mathbb{C})$ at fractional level $k = \frac{p}{u} - 1$ with p, u coprime and positive integers, when the quantum numbers of the highest weight state obey the relations (3.46,3.47). These character formulas generalise character formulas obtained previously in the particular case of fractional level $k = \frac{1}{u} - 1$, and we have emphasized in which ways the generalised formulas are much harder to handle mathematically. The essence of the difficulty is that it is impossible to express them as an infinite product (unlike the case $p = 1$) of modular functions, and as a consequence, they have a non-quasi-periodic behaviour which complicates the derivation of modular transformations as we will discover in the next chapter.

3.4 Summary

In this chapter we have described the $\widehat{sl}(2 \mid 1; \mathbb{C})$ affine algebra. Several properties have been obtained which will be useful in later chapters.

We have discussed different automorphisms between algebras, the most important is the spectral flow which is very helpful in the next chapter.

Chapter 4

Modular behaviour of admissible $\widehat{sl}(2 \mid 1; \mathbb{C})$ characters

4.1 Introduction

In this chapter, we present sumrules involving admissible $\hat{sl}(2)$ as well as the admissible $\widehat{sl}(2 \mid 1; \mathbb{C})$ characters introduced earlier. The structure underlying the sumrules is extremely interesting as a natural generalisation of the situation encountered in two-dimensional matter plus gravity systems. It involves the extension of the sum $\hat{sl}(2)_{k1} \oplus \hat{sl}(2)_{k2}$ of two affine $\hat{sl}(2)$ algebras at*dual* levels k_1 and k_2, i.e. at levels related through $\frac{1}{k_1+2} + \frac{1}{k_2+2} = 1$. The extended algebraic structure is the affine superalgebra $\hat{D}(2|1; k_2)_{k_1}$, which can be decomposed in $\widehat{sl}(2 \mid 1) \oplus \widehat{U}(1)$ or in three $\hat{sl}(2)$ algebras at levels k_1, k_2 and 1. The sumrules we use here are character identities which encode the link between two subalgebra decompositions of $D(2|1; k_2)_{k_1}$ we just mentioned. Schematically we have,

$$\widehat{sl}(2 \mid 1)_{k_1} \oplus \widehat{U}(1) = \hat{sl}(2)_{k1} \oplus \hat{sl}(2)_{k2} \oplus \hat{sl}(2)_1. \tag{4.1}$$

The number of bosonic and fermionic generators in $\widehat{sl}(2 \mid 1; \mathbb{C})$ is the same, so we are faced with a vanishing central charge for $\widehat{sl}(2 \mid 1; \mathbb{C})$, while the $\widehat{U}(1)$ factor, which describes a free scalar theory, contributes one to the central charge. On the right hand side, the sum of central charges is also one thanks to the duality of the levels

k_1 and k_2. Indeed, since the duality implies $k_2 = -\frac{k_1}{k_1+1}$, the central charge is,

$$c = \frac{3k_1}{k_1+2} + \frac{3k_2}{k_2+2} + 1 = 1. \tag{4.2}$$

Sumrules of the same type have been discussed in [66], but only when the level k_1 is fractional of the form $k_1 = \frac{1}{u} - 1$ with $u \in \mathbb{N} \setminus \{1\}$. In that case, the duality of levels implies that $k_2 = u - 1$, i.e. k_2 is an integer so that the sumrules involve, on the right hand side, products of three $\hat{sl}(2)$ characters with two of them at integer level. Here, we consider the more general situation where $k_1 = \frac{p}{u} - 1$, with p, u coprime. Our aim is to exploit the $\hat{sl}(2)$ content of the sumrules to obtain the modular behaviour of admissible $\widehat{sl}(2 \mid 1; \mathbb{C})$ characters for a general value of the parameter p.

In section 2, we show how the various highest weight modules of the different subalgebras organise themselves in the sumrules. We then present in section 3 explicit sumrules in the Ramond sector, which we spectral flow to obtain Neveu-Schwarz sumrules. We have chosen to derive the modular transformations in the Neveu-Schwarz sector in this thesis.

In section 4, we consider the integral form for the sumrules as it will be useful in the calculation of the modular transformations of $\widehat{sl}(2 \mid 1; \mathbb{C})$ admissible characters, which will be presented in section 5. The appendix gives some explicit checks of the general sumrules in the special case p = 1, u = 2, and also provides some technical results relevant for the study of modular transformations.

4.2 Notations and conventions for the sumrules

We consider a class of $\widehat{sl}(2 \mid 1; \mathbb{C})$ representations that can be arrived at as follows. The vertex - operator extension of $\hat{sl}(2)_k \oplus \hat{sl}(2)_{k'}$ where the two $\hat{sl}(2)$ algebras have dual fractional levels, i.e. levels of the form $k = \frac{p}{u} - 1$ and $k' = \frac{u}{p} - 1$ with p, u coprime, provides the affine superalgebra $\hat{D}(2|1;k')_k$ with affine subalgebra $\widehat{sl}(2 \mid 1)_k$. For every $\widehat{sl}(2 \mid 1; \mathbb{C})$ representation L, this vertex - operator extension leads to a decomposition of the sum over the spectral flow orbit of L into a sum of terms $M \otimes M' \otimes M^{(1)}$, where M is an $\hat{sl}(2)_k$ representation, M' is an $\hat{sl}(2)_{k'}$ representation, and $M^{(1)}$ is a representation of the level one $\hat{sl}(2)$ algebra [66] .

More precisely, according to [40], one may write pu sumrules parametrized by

$$1 \leq s' \leq p \qquad \text{and} \qquad 1 \leq s \leq u. \tag{4.3}$$

They are,

$$\bigoplus_{n=s'-s+1-p-u}^{s'-s-1} M_{\frac{n+u-1}{2}-\frac{p}{u}\frac{s-1}{2},\frac{p}{u}-1} \otimes M'_{-\frac{n-s}{2}-\frac{u}{p}\frac{s'-1}{2},\frac{u}{p}-1} \otimes M_{\frac{\epsilon(n+u-1)}{2},1} = \bigoplus_{\theta\in\mathbb{Z}} L_{h_-,h_+,k;\theta} \otimes A_{\frac{u-s-1}{2}-\frac{u}{p}\frac{s-1}{2}-\theta} \tag{4.4}$$

where

$$\epsilon(2n) = 0 \qquad \text{and} \qquad \epsilon(2n+1) = 1 \qquad n \in \mathbb{Z}. \tag{4.5}$$

The modules A_i are Fock modules and the level $k = \frac{p}{u} - 1$ $\widehat{sl}(2 \mid 1;\mathbb{C})$ untwisted highest weight quantum numbers of isospin and hypercharge are given by,

$$\begin{aligned} h_- &= \frac{p-s'}{2} - \frac{p}{u}\frac{s-1}{2} \\ h_+ &= \frac{p-s'}{2} - \frac{p}{u}\frac{s+1}{2}. \end{aligned} \tag{4.6}$$

On the left hand side, start by relabelling $n = r + s' - s - p - u$ so that we obtain,

$$\bigoplus_{r=1}^{p+u-1} M_{\frac{r-1}{2}-\frac{p+u}{u}\frac{s-1}{2},\frac{p}{u}-1} \otimes M'_{\frac{p+u-r-1}{2}-\frac{p+u}{p}\frac{s'-1}{2},\frac{u}{p}-1} \otimes M_{\frac{\epsilon(r+1+p+s+s')}{2},1}, \tag{4.7}$$

where the isospin of the $\hat{sl}(2)_K$ highest weight states are given by,

$$\begin{aligned} j &= \frac{r-1}{2} - \frac{p+u}{u}\frac{s-1}{2} \qquad \text{when} \qquad K = k = \frac{p}{u} - 1 \\ j' &= \frac{p+u-r-1}{2} - \frac{p+u}{p}\frac{s'-1}{2} \qquad \text{when} \qquad K = k' = \frac{u}{p} - 1 \\ j^{(1)} &= \frac{\epsilon(r+1+p+s+s')}{2} \qquad \text{when} \qquad K = 1. \end{aligned} \tag{4.8}$$

We recall here (see chapter 2) that there are $P(U-1)$ irreducible admissible representations of the affine algebra $\hat{sl}(2)$ at level $K = \frac{U}{P} - 2$. The corresponding characters are given by,

$$X^{\hat{sl}(2)}_{R,S,U,P}(x,q) = q^{\frac{J(J+1)}{K+2}+\frac{P}{4U}(1-U^2)} \lambda_{R,S,U,P}(x,q)\,, \tag{4.9}$$

where

$$\lambda_{R,S,U,P}(x,q) =$$
$$x^J \frac{\vartheta_{1,0}(x^U q^{P(R-U)-(S-1)U}, q^{2UP}) - x^{-R} q^{R(S-1)} \vartheta_{1,0}(x^U q^{-P(R+U)-(S-1)U}, q^{2UP})}{\vartheta_{1,1}(x,q)}, \tag{4.10}$$

with $1 \leq R \leq U-1$ and $1 \leq S \leq P$. The isospin of the highest weight state is,

$$J = \frac{R-1}{2} - \frac{S-1}{2}\frac{U}{P}. \tag{4.11}$$

The sumrules involve affine $\hat{sl}(2)$ characters at three different levels. When $K = \frac{p}{u} - 1$, we use the character formula (4.9) with $U = p+u, P = u$, when $K = \frac{u}{p} - 1$, we use the character formula (4.9) with $U = p+u, P = p$ and when $K = 1$, we use $U = 3, P = 1$. The dependence in the angular variable x in (4.10) is different in the three $\hat{sl}(2)$ types of characters. We will write the sumrules explicitly in the next section, with the exact dependence in the angular variables.

4.3 Ramond and Neveu-Schwarz sumrules for $\widehat{sl}(2 \mid 1;\mathbb{C})$

In this section we shall present the character sumrules in the Ramond sector, when the level of the $\widehat{sl}(2 \mid 1;\mathbb{C})$ affine superalgebra is fractional of the form $k = \frac{p}{u} - 1$, with p, u coprime. The original work of this kind is from [66], but it only relates to the special case $p = 1$. As we already stressed, the mathematics is much less complicated in that special case because the character functions are periodic in the spectral flow, leaving us to deal with a finite number of characters transforming among themselves under the modular group. In the general case, the sumrules are still elegant and we show in the appendix how to recover the sumrules in a particularly simple special case where $p = 1$ and $u = 2$ (see appendix C).

We also give here the sumrules in the Neveu-Schwarz sector, by spectral flowing from the Ramond sector according to the prescription given in (3.59). This form of the sumrules is slightly more easy to deal with when studying their modular behaviour. Indeed, Neveu-Schwarz characters transform in Neveu-Schwarz characters under S.

Now let us write down the sumrules in the Ramond sector [40]. We have,

$$\sum_{r=1}^{p+u-1} X^{\hat{sl}(2)_k}_{r,s,p+u,u}(z,q) X^{\hat{sl}(2)_{k'}}_{p+u-r,s',p+u,p}(\zeta^{k+1}y,q) X^{\hat{sl}(2)_1}_{1+\epsilon(p+1+s+s'+r),1,3,1}(\zeta^{-k}y^{-1},q) =$$

$$\sum_{\theta\in Z} X^{\widehat{sl}(2|1;\mathbb{C})_k}_{p+1-s',s,u,p;\theta}(z,\zeta,q)\frac{y^{A(s,s',\theta)}q^{\frac{p}{u}A^2(s,s',\theta)}}{\eta(q)}, \quad (4.12)$$

where

$$A(s,s',\theta) = \frac{u-1-s}{2} - \frac{u}{p}\frac{s'-1}{2} - \theta, \quad (4.13)$$

and where $1 \leq s \leq u$, $1 \leq s' \leq p$, and $k+1 = \frac{p}{u}$, $k'+1 = \frac{u}{p}$. The above expression provides pu sumrules. However, the fundamental range for $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters at level $k = \frac{p}{u} - 1$ should include $2p$ values for the label s', not just p (see (3.50)). It is sufficient to extend the range of s' to $1 \leq s' \leq 2p$ and to recall (see (2.78)) that

$$X^{\hat{sl}(2)'_k}_{p+u-r,s'+p,p+u,p}(\zeta^{k+1}y,q) = X^{\hat{sl}(2)'_k}_{p+u-r,s'-p,p+u,p}(\zeta^{k+1}y,q) = -X^{\hat{sl}(2)'_k}_{r,s',p+u,p}(\zeta^{k+1}y,q). \quad (4.14)$$

We obtain the Neveu-Schwarz sector by flowing

$$\zeta \to \zeta q \qquad \text{and} \qquad y \to yq^{-(k+1)} \quad (4.15)$$

in the sumrules (4.12). Note that this implies we obtain the Neveu-Schwarz $\widehat{sl}(2 \mid 1)$ characters by a different flow to the one used in previous literature, and in particular in Michael Hayes thesis, where the Neveu-Schwarz sector was obtained by flowing in the z variable, namely $z \to z^{-1}q^{-1}$.

So the Neveu-Schwarz sumrules are given by,

$$\sum_{r=1}^{p+u-1} X^{\hat{sl}(2)_k}_{r,s,p+u,u}(z,q)\, X^{\hat{sl}(2)_{k'}}_{p+u-r,s',p+u,p}(\zeta^{k+1}y,q)\, X^{\hat{sl}(2)_1}_{2-\epsilon(p+1+s+s'+r),1,3,1}(\zeta^{-k}y^{-1},q) =$$

$$= q^{\frac{p}{4u}}y^{-1/2}\sum_{\theta\in\mathbb{Z}} X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;\theta+\frac{1}{2}}(z,\zeta,q)\frac{y^{A(s,s',\theta)}q^{\frac{p}{u}[A^2(s,s',\theta)-A(s,s',\theta)]}}{\eta(q)}. \quad (4.16)$$

The above formula is easily established once the following remarks are made. First of all, the only characters on the left hand side of (4.12) which are affected by the flow (4.15) are the $\hat{sl}(2)_1$ characters. They are given by (see appendix B),

$$X^{\hat{sl}(2)}_{1+\epsilon,1,3,1}(\zeta^{-k}y^{-1},q) = \frac{\theta_{\epsilon,1}(\zeta^{-k}y^{-1},q)}{\eta(q)}, \quad (4.17)$$

where the level one theta function is,

$$\theta_{\epsilon,1}(\zeta^{-k}y^{-1},q) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{\epsilon}{2})^2}\zeta^{-k(n+\frac{\epsilon}{2})}y^{-(n+\frac{\epsilon}{2})}. \tag{4.18}$$

Under the flow (4.15) they give,

$$\theta_{\epsilon,1}(\zeta^{-k}y^{-1}q,q) = \zeta^{\frac{k}{2}}y^{\frac{1}{2}}q^{-\frac{1}{4}}\theta_{1+\epsilon,1}(\zeta^{-k}y^{-1},q), \tag{4.19}$$

so that the affine $\hat{sl}(2)_1$ characters transform as,

$$X^{\hat{sl}(2)}_{1+\epsilon(p+1+s+s'+r),1,3,1}(\zeta^{-k}y^{-1}q,q) = \zeta^{\frac{k}{2}}y^{\frac{1}{2}}q^{-\frac{1}{4}}X^{\hat{sl}(2)}_{2-\epsilon(p+1+s+s'+r),1,3,1}(\zeta^{-k}y^{-1},q). \tag{4.20}$$

Second of all, the right hand side of the sumrules (4.12) involves the spectral-flowed $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters,

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;\theta}(z,\zeta q,q) = \zeta^{\frac{k}{2}}q^{\frac{k}{4}}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;\theta+\frac{1}{2}}(z,\zeta,q). \tag{4.21}$$

We are now in a position to discuss the modular behaviour of these $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters.

4.4 Modular Transformations of the Neveu-Schwarz $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters

We first extract an integral representation of the Neveu-Schwarz $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters from the sumrules (4.16) which will be suitable for our study of modular properties. We recall our notations for the various variables entering the sumrules,

$$q = e^{2i\pi\tau},\ z = e^{2i\pi\nu},\ \zeta = e^{2i\pi\mu},\ y = e^{2i\pi\rho}, \tag{4.22}$$

with $Im\tau \geq 0$ and $\nu, \mu, \rho \in \mathbb{C}$.

Multiply left and right of (4.16) by $y^{\frac{1}{2}-A(s,s',\theta)-\theta+n}$ for any integer n, then integrate over a closed contour C around $y = 0$, recalling that,

$$\oint_C dy\ y^m = 2i\pi\delta_{m+1,0}. \tag{4.23}$$

One obtains,

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;n+\frac{3}{2}}(\nu,\mu,\tau) = q^{-\frac{p}{u}[A(s,s',n+1)^2-A(s,s',n+1)+\frac{1}{4}]}\eta(\tau)\int_0^1 d\rho e^{-2i\pi\rho[A(s,s',n+1)-\frac{1}{2}]}$$
$$\times\sum_{r=1}^{p+u-1} X^{\hat{sl}(2)_k}_{r,s,p+u,u}(\nu,\tau)X^{\hat{sl}(2)_{k'}}_{p+u-r,s',p+u,p}((k+1)\mu+\rho,\tau)X^{\hat{sl}(2)_1}_{2-\epsilon(p+1+s+s'+r),1,3,1}(-k\mu-\rho,\tau), \tag{4.24}$$

where $k = \frac{p}{u} - 1$.

The above formula provides a way to derive the modular behaviour of $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters using the well-known modular transformations of $\hat{sl}(2)$ characters. In the case where $p = 1$, the modular transformations of $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters have been calculated in [38], where a decomposition into $\hat{sl}(2)_k$ characters was used, with the branching coefficients being string functions. Examples for $p = 1, u = 2$ and $p = 1, u = 3$ were given in [37].

Under the two transformations S and T generating the modular group, the arguments (ν,μ,τ) of the $\widehat{sl}(2 \mid 1;\mathbb{C})$ transform as,

$$S(\nu,\mu,\tau) = (\frac{\nu}{\tau},\frac{\mu}{\tau},-\frac{1}{\tau}) \qquad T(\nu,\mu,\tau) = (\nu,\mu,\tau+1). \tag{4.25}$$

First consider the modular transformation T. Using the characters and super characters given in the previous chapter, it is quite plain to see that shifting $\tau \to \tau+1$ gives the following results,

$$\begin{aligned} X^R(\tau+1,\nu,\rho) &\to e^{2\pi i\Delta^R}X^R(\tau,\nu,\rho) \\ SX^R(\tau+1,\nu,\rho) &\to e^{2\pi i\Delta^R}SX^R(\tau,\nu,\rho) \\ X^{NS}(\tau+1,\nu,\rho) &\to e^{2\pi i\Delta^{NS}}X^R(\tau,\nu,\rho) \\ SX^{NS}(\tau+1,\nu,\rho) &\to e^{2\pi i\Delta^{NS}}SX^R(\tau,\nu,\rho). \end{aligned} \tag{4.26}$$

where Δ^R is defined by

$$\Delta^R = \frac{h_-^2 - h_+^2}{k+1} + 2\theta h_+ - k\theta^2, \tag{4.27}$$

and the conformal weight for NS sector are given by

$$\theta \to \theta + \frac{1}{2}. \tag{4.28}$$

Now let us discuss the S transform. Unlike in the special case $p = 1$, there is no branching relations of $\widehat{sl}(2 \mid 1; \mathbb{C})$ characters into $\hat{sl}(2)$ characters of the type introduced in [66] (with the branching coefficients being parafermionic characters). We therefore S-transform (4.24), i.e. replace $\tau \to -1/\tau$, $\nu \to \nu/\tau$ and $\mu \to \mu/\tau$, and change the integration variable ρ to ρ/τ. This leads to,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;n+\frac{3}{2}}(\frac{\nu}{\tau}, \frac{\mu}{\tau}, \frac{-1}{\tau}) = \eta(\frac{-1}{\tau}) e^{\frac{2i\pi}{\tau}\frac{p}{u}[A(s,s',n+1)^2 - A(s,s',n+1)+\frac{1}{4}]} \frac{1}{\tau}\int_0^\tau d\rho e^{-\frac{2i\pi\rho}{\tau}[A(s,s',n+1)-1/2]}$$
$$\times \Big\{ \sum_{r=1}^{p+u-1} \chi^{\hat{sl}(2)}_{r,s,p+u,u}(\frac{\nu}{\tau}, \frac{-1}{\tau}) \chi^{\hat{sl}(2)}_{p+u-r,s',p+u,p}(\frac{(k+1)\mu+\rho}{\tau}, \frac{-1}{\tau}) \chi^{\hat{sl}(2)}_{2-\epsilon(p+1+s+s'+r),1,3,1}(\frac{-k\mu-\rho}{\tau}, \frac{-1}{\tau}). \Big\} \quad (4.29)$$

We must now use the S transform of $\hat{sl}(2)$ characters. Recall from Chapter 2 (2.86) that the $\hat{sl}(2)$ characters at level $K = \frac{U}{P} - 2$ transform under S as,

$$\chi^{\widehat{sl(2)}}_{r,s,U,P}(\frac{\nu}{\tau}, \frac{-1}{\tau}) = \frac{1}{2}\sqrt{\frac{2}{UP}} e^{\frac{\pi i \nu^2}{\tau}(\frac{U}{2P}-1)} \times \sum_{r'=1}^{U-1} \sum_{s'=1-P}^{P} e^{i\pi[r(s'-1)+r'(s-1)-(s-1)(s'-1)\frac{U}{P}]} \sin\frac{P\pi r r'}{U} \chi^{\widehat{sl(2)}}_{r',s',U,P}(\nu,\tau). \quad (4.30)$$

Note that in view of (4.30), the above S transformation formula is a rewriting of the more standard relation,

$$\chi^{\widehat{sl(2)}}_{r,s,U,P}(\frac{\nu}{\tau}, \frac{-1}{\tau}) = \sqrt{\frac{2}{UP}} e^{\frac{\pi i \nu^2}{\tau}(\frac{U}{2P}-1)} \times \sum_{r'=1}^{U-1} \sum_{s'=1}^{P} e^{i\pi[r(s'-1)+r'(s-1)-(s-1)(s'-1)\frac{U}{P}]} \sin\frac{P\pi r r'}{U} \chi^{\widehat{sl(2)}}_{r',s',U,P}(\nu,\tau). \quad (4.31)$$

Here, we apply (4.31) for the $\hat{sl}(2)$ at levels corresponding to $(U, P) = (p+u, u), (p+u, p)$ and $(3, 1)$.

We therefore rewrite (4.29) as,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;n+\frac{3}{2}}(\frac{\nu}{\tau}, \frac{\mu}{\tau}, \frac{-1}{\tau}) = (-i)^{\frac{1}{2}} \frac{2\sqrt{2}}{(p+u)\sqrt{3up\tau}} \eta(\tau) e^{\frac{2i\pi}{\tau}\frac{p}{u}[A(s,s',n+1)-\frac{1}{2}]^2}$$
$$\times \int_0^\tau d\rho e^{-\frac{2i\pi\rho}{\tau}[A(s,s',n+1)-\frac{1}{2}]} e^{-\frac{i\pi(u-p)\nu^2}{2u\tau}} e^{\frac{i\pi(u-p)}{2p\tau}[\frac{p}{u}\mu+\rho]^2} e^{\frac{i\pi}{2\tau}[\frac{u-p}{u}\mu-\rho]^2} \sigma(\nu,\mu,\rho,\tau), \quad (4.32)$$

with

$$\sigma(\nu,\mu,\rho,\tau)=\sum_{r=1}^{p+u-1}\sum_{r''=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{r'''=1}^{p+u-1}\sum_{s'''=1}^{p}\sum_{r^{iv}=1}^{2}$$
$$e^{i\pi[r(s''-1)+r''(s-1)-(s-1)(s''-1)\frac{p+u}{u}+(p+u-r)(s'''-1)+(s'-1)r'''-(s'-1)(s'''-1)\frac{p+u}{p}]}$$
$$\times\sin\frac{\pi urr''}{p+u}\sin\frac{\pi p(p+u-r)r'''}{p+u}\sin\frac{\pi(2-\epsilon(p+1+s+s'+r)r^{iv}}{3}$$
$$\times\chi^{\hat{sl}(2)}_{r'',s'',p+u,u}(\nu,\tau)\chi^{\hat{sl}(2)}_{r''',s''',p+u,p}(\frac{p}{u}\mu+\rho,\tau)\chi^{\hat{sl}(2)}_{r^{iv},1,3,1}(\frac{u-p}{u}\mu-\rho,\tau)\quad(4.33)$$

After some lengthy calculations described in the appendix D, we can rewrite the above as, (**from now on, we consider $p+u$ even, which implies p and u are odd since they are coprime**),

$$\sigma(\nu,\mu,\rho,\tau)=\frac{\sqrt{3}}{4}(p+u)\frac{1}{\eta(\tau)}\sum_{s''=1}^{u}\sum_{s'''=1}^{2p}e^{i\pi[ss'''+s's''-(s-1)(s''-1)\frac{p}{u}-(s'-1)(s'''-1)\frac{u}{p}]}$$
$$\times\sum_{\theta\in\mathbb{Z}}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'',u,p;\theta+\frac{1}{2}}(\nu,\mu,\tau)y^{A(s'',s''',\theta)-\frac{1}{2}}q^{\frac{p}{u}[A(s'',s''',\theta)-\frac{1}{2}]^2}.\quad(4.34)$$

The S transform of the Neveu-Schwarz characters takes the form,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;n+\frac{3}{2}}(\frac{\nu}{\tau},\frac{\mu}{\tau},\frac{-1}{\tau})=\frac{(-i)^{\frac{1}{2}}}{\sqrt{2up\tau}}e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}}e^{\frac{2i\pi}{\tau}\frac{p}{u}[A(s,s',n+1)-\frac{1}{2}]^2}$$
$$\times\sum_{s''=1}^{u}\sum_{s'''=1}^{2p}\sum_{\theta\in\mathbb{Z}}e^{i\pi[ss'''+s's''-(s-1)(s''-1)\frac{p}{u}-(s'-1)(s'''-1)\frac{u}{p}]}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'',u,p;\theta+\frac{1}{2}}(\nu,\mu,\tau)$$
$$\times q^{\frac{p}{u}[A(s'',s''',\theta)-\frac{1}{2}]^2}\int_0^{\tau}d\rho\; e^{-\frac{2i\pi\rho}{\tau}[A(s,s',n+1)-\frac{1}{2}]}e^{\frac{i\pi u}{2p\tau}\rho^2}e^{2i\pi\rho[A(s'',s''',\theta)-\frac{1}{2}]}.\quad(4.35)$$

In the remaining part of this chapter, we attempt to rewrite the above modular transformation in a more suggestive way.

First set $\theta=um+\beta$, for $\beta=0,...,u-1$ and $m\in\mathbb{Z}$, noting that,

$$A(s'',s''',um+\beta)\quad=\quad A(s'',s'''+2pm,\beta)$$

and

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'',u,p;um+\beta+\frac{1}{2}}(\nu,\mu,\tau)\quad=\quad X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s'''-2pm,s'',u,p;\beta+\frac{1}{2}}(\nu,\mu,\tau)\quad(4.36)$$

according to the definition (4.13) and the properties (3.53). Then, relabel $s'''+2pm=$

m' for $m' \in \mathbb{Z}$, and write,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s',s,u,p;n+\frac{3}{2}}(\frac{\nu}{\tau},\frac{\mu}{\tau},\frac{-1}{\tau}) = \frac{(-i)^{\frac{1}{2}}}{\sqrt{2up\tau}} e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}} e^{\frac{2i\pi}{\tau}\frac{p}{u}[A(s,s',n+1)-\frac{1}{2}]^2}$$
$$\times \sum_{s''=1}^{u}\sum_{m'\in\mathbb{Z}}\sum_{\beta=0}^{u-1} e^{i\pi[sm'+s's''-(s-1)(s''-1)\frac{p}{u}-(s'-1)(m'-1)\frac{u}{p}]}\chi^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-m',s'',u,p;\beta+\frac{1}{2}}(\nu,\mu,\tau)$$
$$\times\, q^{\frac{p}{u}[A(s'',m',\beta)-\frac{1}{2}]^2}\int_0^\tau d\rho\; e^{-\frac{2i\pi\rho}{\tau}[A(s,s',n+1)-\frac{1}{2}]} e^{\frac{i\pi u}{2p\tau}\rho^2} e^{2i\pi\rho[A(s'',m',\beta)-\frac{1}{2}]}. \quad (4.37)$$

Our next task is to reexpress the ρ–integral in terms of the function L defined in (2.92). After some elementary manipulations, we obtain,

$$\frac{1}{2p}\int_0^\tau d\rho\; e^{-\frac{2i\pi\rho}{\tau}[A(s,s',n+1)-\frac{1}{2}]} e^{\frac{i\pi u}{2p\tau}\rho^2} e^{2i\pi\rho[A(s'',m',\beta)-\frac{1}{2}]} =$$
$$e^{-\frac{2i\pi p}{u\tau}[A(s,s',n+1)-\frac{1}{2}-\tau(A(s'',m',\beta)-\frac{1}{2})]^2} L^{(u)}_{-p(2\beta+2+s'')-u(m'-1-p)}(\frac{\tau}{2pu},\frac{s+2n+4-u}{2u}+\frac{s'-1}{2p}). \quad (4.38)$$

Relabelling $s' = p+1-r$ and $m' = p+1-m$, the S transform of the $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters becomes,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;n+\frac{3}{2}}(\frac{\nu}{\tau},\frac{\mu}{\tau},\frac{-1}{\tau}) = -(-i)^{\frac{1}{2}}\sqrt{\frac{2p}{u\tau}}\, e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}}$$
$$\times \sum_{s''=1}^{u}\sum_{m\in\mathbb{Z}}\sum_{\beta=0}^{u-1} e^{i\pi(r-m)} e^{2i\pi[(\beta+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}}$$
$$\times\, \chi^{\widehat{sl}(2|1;\mathbb{C})}_{m,s'',u,p;\beta+\frac{1}{2}}(\nu,\mu,\tau) L^{(u)}_{-p(2\beta+2+s'')+um}(\frac{\tau}{2pu},\frac{s+2n+4}{2u}-\frac{r}{2p}). \quad (4.39)$$

We now rewrite $\chi^{\widehat{sl}(2|1;\mathbb{C})}_{m,s'',u,p;\beta+\frac{1}{2}}(\nu,\mu,\tau)$ for any integer value m in terms of $\chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;n+\frac{3}{2}}(\nu,\mu,\tau)$ in an effort to eliminate the m-dependence in the $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters. Although it is a rather non-trivial exercise, one may check the following relations using the explicit expressions for twisted $\widehat{sl}(2 \mid 1;\mathbb{C})$ characters (3.52) when

$\theta = \beta + \frac{1}{2}$,

$$\begin{aligned}
X^{\widehat{sl}(2|1;\mathbb{C})}_{m,s'',u,p,;\beta+\frac{1}{2}}(\nu,\mu,\tau) &= (-1)^{r-m} X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s'',u,p,;\beta+\frac{1}{2}}(\nu,\mu,\tau) \\
&+ (-1)^{1+r-m} z^{-\frac{1}{2}} q^{-\frac{1}{4}-\frac{up}{4}} \frac{\vartheta_{1,0}(z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta^3(q)} z^{\frac{r-1}{2}-\frac{s''-1}{2}\frac{p}{u}} \\
&\times \zeta^{\frac{r-1}{2}-(\beta+\frac{3}{2}+\frac{s''-1}{2})\frac{p}{u}} q^{-(\beta+\frac{3}{2})^2\frac{p}{u}+(\beta+\frac{3}{2})(r-1-(s''-1)\frac{p}{u})}
\end{aligned}$$

$$\times \begin{cases} \sum_{a=0}^{r-m-1}(-1)^a(\zeta z)^{-\frac{1}{2}a} q^{-(\beta+\frac{3}{2})a}\Lambda_{r-1-a,s'',u,p}(\nu,\tau), & \text{for} \quad r-m>0, \\ \sum_{a=r-m}^{-1}(-1)^{1+a}(\zeta z)^{-\frac{1}{2}a} q^{-(\beta+\frac{3}{2})a}\Lambda_{r-1-a,s'',u,p}(\nu,\tau), & \text{for} \quad r-m<0, \end{cases} \tag{4.40}$$

where we have defined the function,

$$\Lambda_{r,s,u,p}(\nu,\tau) = \vartheta_{1,0}(z^p q^{-p(s-1+u)+ur}, q^{2up}) - q^{r(s-1)} z^{-r} \vartheta_{1,0}(z^p q^{-p(s-1+u)-ur}, q^{2up}), \tag{4.41}$$

and note the following property for future reference,

$$\Lambda_{r-2pj,s,u,p}(\nu,\tau) = z^{pj} q^{-upj^2} q^{-pj(s-1)+ujr} \Lambda_{r,s,u,p}(\nu,\tau), \qquad j \in \mathbb{Z}. \tag{4.42}$$

Remark: as an intermediate step in obtaining the above, one may check that,

$$\begin{aligned}
\psi_{r,s,u,p}(\nu,\mu,\tau) &= (-1)^{r-m}(z\zeta q^2)^{-\frac{1}{2}(r-m)}\psi_{m,s,u,p}(\nu,\mu,\tau) \\
&+ \begin{cases} \sum_{a=0}^{r-m-1}(-1)^a(\zeta z)^{-\frac{1}{2}a} q^{-a-\frac{up}{4}}\Lambda_{r-1-a,s,u,p}(\nu,\tau), & \text{for} \quad r-m>0, \\ \sum_{a=r-m}^{-1}(-1)^{1+a}(\zeta z)^{-\frac{1}{2}a} q^{-a-\frac{up}{4}}\Lambda_{r-1-a,s,u,p}(\nu,\tau), & \text{for} \quad r-m<0, \end{cases}
\end{aligned} \tag{4.43}$$

using the expression (3.49). Inserting (4.40) in (4.39), we obtain,

$$\chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s,u,p;n+\frac{3}{2}}(\frac{\nu}{\tau},\frac{\mu}{\tau},\frac{-1}{\tau}) = E(1) + E(2), \tag{4.44}$$

where

$$\begin{aligned}
E(1) &= -(-i)^{\frac{1}{2}}\sqrt{\frac{2p}{u\tau}} e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}} \sum_{s''=1}^{u}\sum_{\beta=0}^{u-1} e^{2i\pi[(\beta+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}} \\
&\times X^{\widehat{sl}(2|1;\mathbb{C})}_{r,s'',u,p;\beta+\frac{1}{2}}(\nu,\mu,\tau)\{\sum_{m\in\mathbb{Z}} L^{(u)}_{-p(2\beta+2+s'')+um}(\frac{\tau}{2pu},\frac{s+2n+4}{2u}-\frac{r}{2p})\}
\end{aligned} \tag{4.45}$$

and

$$E(2) = (-i)^{\frac{1}{2}}\sqrt{\frac{2p}{u\tau}}e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}}q^{-\frac{1}{4}-\frac{up}{4}}\frac{\vartheta_{1,0}(z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta^3(q)}$$
$$\times\sum_{s''=1}^{u}\{\sum_{m=-\infty}^{r-1}\sum_{a=0}^{r-m-1}-\sum_{m=r+1}^{\infty}\sum_{a=r-m}^{-1}\}\sum_{\beta=0}^{u-1}(-1)^a e^{2i\pi[(\beta+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}}$$
$$\times z^{\frac{r-2-a}{2}-\frac{s''-1}{2}\frac{p}{u}}\zeta^{\frac{r-1-a}{2}-(\beta+\frac{3}{2}+\frac{s''-1}{2})\frac{p}{u}}q^{-(\beta+\frac{3}{2})^2\frac{p}{u}+(\beta+\frac{3}{2})(r-1-a-(s''-1)\frac{p}{u})}$$
$$\Lambda_{r-1-a,s'',u,p}(\nu,\tau)L^{(u)}_{-p(2\beta+2+s'')+um}(\frac{\tau}{2pu},\frac{s+2n+4}{2u}-\frac{r}{2p}). \quad (4.46)$$

It is easy to see, from the definition (2.92), that

$$\sum_{m\in\mathbb{Z}}L^{(u)}_{-p(2\beta+2+s'')+um}(\frac{\tau}{2pu},\frac{s+2n+4}{2u}-\frac{r}{2p}) = i^{\frac{1}{2}}\sqrt{\frac{\tau}{2up}}, \quad (4.47)$$

so that $E(1)$ is given by,

$$E(1) = -\frac{1}{u}e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}}\sum_{s''=1}^{u}\sum_{\beta=0}^{u-1}e^{2i\pi[(\beta+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}}\chi^{\widehat{sl}(2|1;\mathbb{C})}_{r,s'',u,p;\beta+\frac{1}{2}}(\nu,\mu,\tau). \quad (4.48)$$

The treatment of $E(2)$ is more involved. First we flip the sign of m, then relabel $m = m' - r$ and $a = a' - 1$. This implies in particular,

$$\{\sum_{m=-\infty}^{r-1}\sum_{a=0}^{r-m-1}-\sum_{m=r+1}^{\infty}\sum_{a=r-m}^{-1}\}\to\{\sum_{a'=1}^{\infty}\sum_{m'=a}^{\infty}-\sum_{a'=-\infty}^{0}\sum_{m'=-\infty}^{a'-1}\}. \quad (4.49)$$

We also use (2.120) to rewrite, ($\eta = \frac{s+2n+4}{2u} - \frac{r}{2p}$),

$$\begin{aligned}\sum_{m'=a'}^{\infty}L^{(u)}_{-p(2\beta+2+s'')-u(m'-r)}(\frac{\tau}{2pu},\eta) &= L^{-}_{-p(2\beta+2+s'')+u(r+1-a')}(\frac{\tau}{2pu},\eta)\\ \sum_{m'=-\infty}^{a'-1}L^{(u)}_{-p(2\beta+2+s'')-u(m'-r)}(\frac{\tau}{2pu},\eta) &= L^{+}_{-p(2\beta+2+s'')+u(r+1-a')}(\frac{\tau}{2pu},\eta), \quad (4.50)\end{aligned}$$

and therefore,

$$E(2) = (-i)^{\frac{1}{2}}\sqrt{\frac{2p}{u\tau}}e^{i\pi\frac{(u-p)(\mu^2-\nu^2)}{2u\tau}}q^{-\frac{1}{4}-\frac{up}{4}}\frac{\vartheta_{1,0}(z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta^3(q)}$$
$$\times\sum_{s''=1}^{u}\sum_{\beta=0}^{u-1}\{\sum_{a'=-\infty}^{0}L^{+}_{-p(2\beta+2+s'')+u(r+1-a')}-\sum_{a'=1}^{\infty}L^{-}_{-p(2\beta+2+s'')+u(r+1-a')}\}(\frac{\tau}{2pu},\eta)$$
$$\times(-1)^{a'}e^{2i\pi[(\beta+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}}z^{\frac{r-1-a'}{2}-\frac{s''-1}{2}\frac{p}{u}}\zeta^{\frac{r-a'}{2}-(\beta+\frac{3}{2}+\frac{s''-1}{2})\frac{p}{u}}$$
$$\times q^{-(\beta+\frac{3}{2})^2\frac{p}{u}+(\beta+\frac{3}{2})(r-a'-(s''-1)\frac{p}{u})}\Lambda_{r-a',s'',u,p}(\nu,\tau). \quad (4.51)$$

Now split a' modulo $2pj$, $a' = 2pj - a''$, and use (4.42),

$$
\begin{aligned}
E(2) = (-i)^{\frac{1}{2}} \sqrt{\frac{2p}{u\tau}} e^{i\pi \frac{(u-p)(\mu^2-\nu^2)}{2u\tau}} q^{-\frac{1}{4}-\frac{up}{4}} \frac{\vartheta_{1,0}(z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta^3(q)} \\
\times \sum_{s''=1}^{u} \sum_{\beta=0}^{u-1} \sum_{a''=0}^{2p-1} \Big\{ \sum_{j=-\infty}^{0} L^{+}_{-p(2\beta+2uj+2+s'')+u(r+1+a'')} - \sum_{j=1}^{\infty} L^{-}_{-p(2\beta+2uj+2+s'')+u(r+1+a'')} \Big\} (\frac{\tau}{2pu},\eta) \\
\times (-1)^{a''} e^{2i\pi[(\beta+uj+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}} z^{\frac{r-1+a''}{2}-\frac{s''-1}{2}\frac{p}{u}} \zeta^{\frac{r+a''}{2}-(\beta+uj+\frac{3}{2}+\frac{s''-1}{2})\frac{p}{u}} \\
\times q^{-(\beta+uj+\frac{3}{2})^2\frac{p}{u}+(\beta+uj+\frac{3}{2})(r+a''-(s''-1)\frac{p}{u})} \Lambda_{r+a'',s'',u,p}(\nu,\tau). \quad (4.52)
\end{aligned}
$$

Relabelling $\beta + uj = j' + u$, we may write,

$$
\begin{aligned}
E(2) = (-i)^{\frac{1}{2}} \sqrt{\frac{2p}{u\tau}} e^{i\pi \frac{(u-p)(\mu^2-\nu^2)}{2u\tau}} q^{-\frac{1}{4}-\frac{up}{4}} \frac{\vartheta_{1,0}(z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)\vartheta_{1,0}(z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-\frac{1}{2}},q)}{\vartheta_{1,1}(z,q)\eta^3(q)} \\
\times \sum_{s''=1}^{u} \sum_{a''=0}^{2p-1} \Big\{ \sum_{j'=-\infty}^{-1} L^{+}_{-p(2j'+2+s'')+u(r+1+a''-2p)} - \sum_{j'=0}^{\infty} L^{-}_{-p(2j'+2+s'')+u(r+1+a''-2p)} \Big\} (\frac{\tau}{2pu},\eta) \\
\times (-1)^{a''} e^{2i\pi[(j'+\frac{3}{2})(s+2n+4)+(n+\frac{5}{2})(s''-1)]\frac{p}{u}} z^{\frac{r-1+a''}{2}-\frac{s''-1}{2}\frac{p}{u}} \zeta^{\frac{r+a''}{2}-(j'+u+\frac{3}{2}+\frac{s''-1}{2})\frac{p}{u}} \\
\times q^{-(j'+u+\frac{3}{2})^2\frac{p}{u}+(j'+u+\frac{3}{2})(r+a''-(s''-1)\frac{p}{u})} \Lambda_{r+a'',s'',u,p}(\nu,\tau). \quad (4.53)
\end{aligned}
$$

4.5 Summary

In this chapter we have considered sumrules for the Ramond and Neveu - Schwarz sectors of $\widehat{sl}(2|1;\mathbb{C})_k$ at level $k = \frac{p}{u} - 1$. The approach used to examine this problem is that of studying the decomposition of $\widehat{sl}(2|1)_k$, in $\widehat{sl}(2)_k$, characters. This gives rise to sumrules involving triple products of characters $\widehat{sl}(2)_k$, $\widehat{sl}(2)_{k'}$ and $\widehat{sl}(2)_1$ with $(k+1)(k'+1) = 1$.

The expressions for the sumrules in the case $p \neq 1$ are vastly more complicated than for the case of $p = 1$, which was studied in [66]. We have been able to determine consistent modular transformation for $k = \frac{p}{u} - 1$ and found these to agree with the modular transformations found in the $(p = 1)$ case [37] [38].

Chapter 5

Conclusions

In any physical context where two - dimensional conformal field theory or any extension thereof is relevant, a good understanding of the representation theory of the underlying infinite - dimensional algebra is valuable.

In string theory so far, unitary representations have been extensively studied and used, especially those corresponding to rational theories. In these theories, the physical states organize themselves in a finite number of unitary irreducible highest weight state representations, whose corresponding characters are building blocks of the partition functions. The modular invariance of the latter is central for the consistency of those theories, and therefore, it is essential to study the behaviour of characters under the modular group. A lot is known about this behaviour as long as one deals with rational theories, like the minimal unitary representations of conformal and superconformal theories or the integrable Kac - Moody representations in the context of the WZNW models. So far, it is mainly in these contexts that the physical interpretation in manageable . However, there are important applications, in non - critical string theory and also in condensed matter physics, where non - unitary representations are needed.

The motivation of this thesis was originally in string theory, where gauged G/G WZNW models provide a method to tackle the study of strings coupled to $2d$ gravity or supergravity. In this area, affine Kac - Moody algebras or superalgebras $\hat{G}$ at fractional level play an important rôle, and it soon became desirable to deepen our knowledge of their representation theory and corresponding characters.

Our work, which is quite mathematical in nature, is a study of the behaviour under modular transformations of the minimal non - unitary $N=2$ superconformal characters as well as of the fractional level non - unitary admissible characters of the affine superalgebra $\widehat{sl}(2|1)$.

Our approach has been pedestrian and very concrete. It has the merit of providing reliable checks of more sophisticated techniques which were developed simultaneously by Taormina, Semikhatov and Tipunin, and which involve the use of Appell functions [68].

Although we have concentrated our efforts on two specific algebras here, we believe there is a general pattern which will emerge and which might even provide a new insight in the study of strings compactified on a K_3 manifold where the $N=4$ superconformal algebra is crucial.

Here follows a summary of our results. In Chapter 2, we started by studying the $N=2$ superconformal algebra at central charge $c=3(1-\frac{p}{u})$, $(p,u)=1$, $u\in\mathbb{N}$, $p\in\mathbb{Z}^*$. We used the general character formula of [60] and applied a spectral flow with integer parameter θ to it, generating an infinite family of so - called " Ramond sector "characters.

In the case where $p=1$, the above characters are quasi - periodic in the spectral flow and can be written as products of theta functions. We show the link between these characters and the unitary minimal $N=2$ characters previously given in the literature . Although their modular transformations have been known for a long time [41] [42], we rederived them to emphasize how much more easily they can be obtained than in the case of general p. In the latter case, we used a remarkable branching rule between affine $\widehat{sl}(2)$ and $N=2$ characters due to [60]. The analysis requires quite a bit of work but we managed to obtain a formula for the S transform which splits in two terms. The second term vanishes when p is set to one while the first term yields the S transform of the N $=2$ unitary minimal characters. The interpretation of the second term when $p\neq 1$ is still under intense investigation at present.

In Chapter 3, we introduce the $\widehat{sl}(2|1)$ affine superalgebra. By making use of data provided by Taormina and Semikhatov, we explicitly established a dictionary between their notations and the conventions used in previous literature in the particular case where the level of $\widehat{sl}(2|1)$ is $k+1=\frac{p}{u}$ with $p=1$. Although the characters may be written as products of generalized theta functions when $p=1$, they loose this property in general, exactly for the same reason as the $N=2$ non - unitary minimal characters do : they are non quasi - periodic in the spectral flow.

Chapter 4 is the truly original part of this thesis. Its objective is to derive the modular transformations of $\widehat{sl}(2|1)$ admissible characters at fractional level $k+1=\frac{p}{u}$, $p \neq 1$.

We used a generalization to $p \neq 1$ of the sumrules presented in [66] and which involve, besides the $\widehat{sl}(2|1)_k$ characters, a sum of triple products of $\widehat{sl}(2)$ characters at levels k, k', and 1 with $(k+1)(k'+1)=1$.

The sumrules provide an integral representation of $\widehat{sl}(2|1)_k$ characters which we explicit in order to achieve our goal. As for $N=2$, the S modular transformation here can be separated into two terms, one of which vanishes at $p=1$. It is extremely satisfactory to be able to check that the first term is in agreement with [38] when p is set to one. Here again, the structure of the second term has not been completely decoded and will only be clear once the derivation via Appell functions is available to us.

Appendix A

Behaviour of F and Theta function Under Shift

In this appendix we show the behaviour of the F and Theta functions under spectral flow (in Chapter 3). This will be useful for deriving a relationship between the general character and class V and class IV character. As we said in chapter 3, we shall calculate F^R and $\theta_{1,1}\theta_{1,0}$ under following shift:

$$\zeta \to \zeta q^{\alpha u}, \qquad z \to z q^{-\alpha u}. \tag{A.0.1}$$

For this purpose first of all we consider $F(q,z,\zeta)$ as follows,

$$F(q,z,\zeta) = \frac{\prod_{n=1}^{\infty}(1+z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^n)(1+z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{n-1})(1+z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^n)}{(1+z^{-\frac{1}{2}}q^{n-1})(1-zq^n)(1-q^n)^2(1-z^{-1}q^{n-1})}. \tag{A.0.2}$$

By using the above shift, we have

$$\begin{aligned} F(q,zq^{-\alpha u},\zeta q^{\alpha u}) &= \prod_{n=1}^{\infty}(1+z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^n)(1+z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{n-1})(1+z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{n-\alpha u}) \\ &\quad \times \frac{(1+z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{n-1+\alpha u})}{(1+z^{-1}q^{n-1+\alpha u})(1-zq^{n-\alpha u})(1-q^n)^2}, \end{aligned} \tag{A.0.3}$$

and

$$\begin{aligned} F(q,zq^{-\alpha u},\zeta q^{\alpha u}) &= \prod_{n=1}^{\infty}(1+z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^n) \\ &\quad \times \frac{(1+z^{-\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{n-1})(1+z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^n)(1+z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{n-1})}{(1-q^n)^2(1-zq^n)(1-z^{-1}q^{n-1})} \\ &\quad \times \frac{(1+z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{1-\alpha u})...(1+z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^0)}{(1+z^{-\frac{1}{2}}\zeta^{\frac{1}{2}})...(1+z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{\alpha u-1})}\frac{(1-z^{-1})...(1-z^{-1}q^{\alpha u-1})}{(1+z^1)...(1+z^1q^{-\alpha u})}. \end{aligned} \tag{A.0.4}$$

So finally, we will arrive at

$$F(q, zq^{-\alpha u}, \zeta q^{\alpha u}) = z^{-\frac{\alpha u}{2}} \zeta^{-\frac{\alpha u}{2}} (-1)^{\alpha u} F(q, z, \zeta). \tag{A.0.5}$$

In here we want to calculate the theta function under the above shift,

$$\theta_{1,1}(q^u, zq^{M+M'+2-\alpha u}) = \prod_{n=1}^{\infty}(1 - z^{-1}q^{u(n-1)-M-M'-2+\alpha u})(1 - zq^{un+M+M'+2-\alpha u})(1 - q^n), \tag{A.0.6}$$

So

$$\begin{aligned}\theta_{1,1}(q^u, zq^{M+M'+2-\alpha u}) &= \prod_{n=1}^{\infty}(1 - z^{-1}q^{u(n-1)-M-M'-2})(1 - zq^{un+M+M'+2})(1 - q^n) \\ &\times \frac{(1 - zq^{u+M+M'+2-\alpha u})...(1 - zq^{M+M'+2})}{(1 - z^{-1}q^{-M-M'-2})...(1 - z^{-1}q^{-M-M'-2+\alpha u-u})},\end{aligned} \tag{A.0.7}$$

Finally we have ;

$$\theta_{1,1}(q^u, zq^{M+M'+2-\alpha u}) = (-1)^{\alpha} z^{\alpha} q^{\alpha(M+M'+2)} q^{-\frac{u}{2}\alpha(\alpha-1)} \theta_{1,1}(q^u, zq^{M+M'+2}). \tag{A.0.8}$$

For $\theta_{1,0}$, we can write as

$$\begin{aligned}&\theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{M'+1})\theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{M+1-\alpha u}) = \theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{M'+1}) \\ &\times \prod_{n=1}^{\infty}(1 + z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{u(n-1+\alpha)-M-1})(1 + z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{u(n-\alpha)+M+1})(1 - q^n),\end{aligned} \tag{A.0.9}$$

$$\begin{aligned}&\theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{M'+1})\theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{M+1-\alpha u}) = \theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{M'+1}) \\ &\times \prod_{n=1}^{\infty}(1 + z^{\frac{-1}{2}}\zeta^{\frac{1}{2}}q^{u(n-1)-M-1})(1 + z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{un+M+1})(1 - q^n) \\ &\times \frac{(1 + z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{u(1-\alpha)+M+1})...(1 + z^{\frac{1}{2}}\zeta^{-\frac{1}{2}}q^{M+1})}{(1 + z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-M-1})..(1 + z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{-M-1+u\alpha-u})} \\ &= z^{\frac{\alpha}{2}}\zeta^{-\frac{\alpha}{2}}q^{\alpha(M+1)}q^{-\frac{\alpha u}{2}(\alpha-1)}\theta_{1,0}(q^u, z^{\frac{1}{2}}\zeta^{\frac{1}{2}}q^{M'+1})\theta_{1,0}(q^u, z^{-\frac{1}{2}}\zeta^{\frac{1}{2}}q^{M+1})\end{aligned} \tag{A.0.10}$$

Appendix B

Theta function and Level one Character

The $\widehat{sl(2)}$ characters at level $k = 1$ can be expressed in terms of theta functions [66],

$$X^{s\hat{l}(2)}_{1+\epsilon(p+1+s+s'+r),1,3,1}(q, \zeta^{-k} y^{-1} q) = \frac{\theta_{\epsilon(p+1+s+s'+r),1}(q, \zeta^{-k} y^{-1} q)}{\eta(q)}. \tag{B.0.1}$$

For theta function we can write as,

$$\theta_{\epsilon(p+1+s+s'+r),1}(q, \zeta^{-k} y^{-1} q) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{\epsilon(p+1+s+s'+r)}{2})^2} (\zeta^{-k} y^{-1})^{(n+\frac{\epsilon(p+1+s+s'+1)}{2})}$$

So we can write

$$\begin{aligned} \theta_{\epsilon(p+1+s+s'+r),1}(q, \zeta^{-k} y^{-1} q) &= q^{\epsilon(p+1+s+s'+r)+\frac{\epsilon(p+1+s+s'+r)}{2}} \zeta^{-k\frac{\epsilon(p+1+s+s'+r)}{2}} y^{-\frac{\epsilon(p+1+s+s'+r)}{2}} \\ &\times \sum_{n\in\mathbb{Z}} q^{n^2+n\epsilon(p+1+s+s'+r)+n} (\zeta^{-k} y^{-1})^n. \end{aligned} \tag{B.0.2}$$

The above expression restricts us to write the following equation;

$$\begin{aligned} \theta_{\epsilon(p+1+s+s'+r),1}(q, \zeta^{-k} y^{-1} q) &= q^{3\frac{\epsilon(p+1+s+s'+r)}{4}} \zeta^{-k\frac{\epsilon(p+1+s+s'+r)}{2}} y^{-\frac{\epsilon(p+1+s+s'+r)}{2}} \\ &\times q^{-\frac{1}{4}(1+\epsilon(p+1+s+s'+r))^2} (\zeta^{-k} y^{-1})^{-(1+\epsilon(p+1+s+s'+r))} \\ &\times \sum_{n\in\mathbb{Z}} q^{(n+\frac{1+\epsilon(p+1+s+s'+r)}{2})^2} (\zeta^{-k} y^{-1})^{(n+\frac{1+\epsilon(p+1+s+s'+r)}{2})^2}. \end{aligned} \tag{B.0.3}$$

Finally we will arrive at

$$\theta_{\epsilon(p+1+s+s'+r),1}(q,\zeta^{-k}y^{-1}q) = q^{-\frac{1}{4}}y^{\frac{1}{2}}\zeta^{\frac{k}{2}}\theta_{\epsilon(p+1+s+s'+r),1}(q,\zeta^{-k}y^{-1}). \tag{B.0.4}$$

By using the relation between theta function and level one character we can write

$$X^{s\hat{l}(2)}_{1+\epsilon(p+1+s+s'+r),1,3,1}(q,\zeta^{-k}y^{-1}q) = q^{-\frac{1}{4}}y^{\frac{1}{2}}\zeta^{\frac{k}{2}}X^{s\hat{l}(2)}_{2+\epsilon(p+1+s+s'+r),1,3,1}(q,\zeta^{-k}y^{-1}) \tag{B.0.5}$$

Appendix C

Link Between General Character And [66]

We only tested of $p = 1$, $u = 2$ [66], [37] with our formula. Now, we would like to compare this general formula with sumrules given in [66], where $p = 1$.

Let us first look at $B^R_\lambda(\tau, \sigma, \nu, \rho)$. The check is a bit involved in general and we will only test the theory $p = 1$, $u = 2$. In that paper , the sumrule with $\lambda = 0$ is as follows,

$$\begin{aligned} B^R_0 &= \theta_{1,1}(\tau, \frac{\nu}{2} - \rho) X^{sl(2)}_{3,2,1,1}(\tau, \sigma) X^{sl(2)}_{3,1,2,1}(\tau, \frac{\nu}{2} + \rho) \\ &+ \theta_{0,1}(\tau, \frac{\nu}{2} - \rho) X^{sl(2)}_{3,2,2,1}(\tau, \sigma) X^{sl(2)}_{3,1,1,1}(\tau, \frac{\nu}{2} + \rho). \end{aligned} \tag{C.0.1}$$

And in terms of class V and IV, we have

$$A^R_0 = \theta_{2,1}(\tau, -\rho) X^{R,IV}_{0,0}(\tau, \sigma, \rho) + \theta_{0,2}(\tau, -\rho) X^{R,V}_{0,0}(\tau, \sigma, \nu) \tag{C.0.2}$$

By using the following formulas,

$$\begin{aligned} \frac{1}{\eta(q)} \theta_{1,1}(\tau, \frac{\nu}{2} - \rho) &= X^{sl(2)}_{3,1,2,1}(\tau, \frac{\nu}{2} - \rho) \\ \frac{1}{\eta(q)} \theta_{0,1}(\tau, \frac{\nu}{2} - \rho) &= X^{sl(2)}_{3,1,1,1}(\tau, \frac{\nu}{2} - \rho) \end{aligned} \tag{C.0.3}$$

and

$$\begin{aligned} X^{R,IV}_{0,0}(\tau, \sigma, \nu) &= X^{sl(2/1);R}_{\frac{1}{2},0,-\frac{1}{2};1}(\tau, \sigma, \nu) \\ X^{R,V}_{0,0}(\tau, \sigma, \nu) &= X^{sl(2/1);R}_{\frac{1}{2},0,-\frac{1}{2};0}(\tau, \sigma, \nu) \end{aligned} \tag{C.0.4}$$

So finally the sumrule $\lambda = 0$ in [66] can be re - written as:

$$X^{sl(2)}_{3,2,1,1}(\tau,\sigma)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}-\rho)+$$
$$X^{sl(2)}_{3,2,2,1}(\tau,\sigma)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}-\rho)=$$
$$\frac{1}{\eta(\tau)}\theta_{2,2}(\tau,-\rho)X^{sl(2/1);R}_{\frac{1}{2},0;1}(\tau,\sigma,\nu)+$$
$$\frac{1}{\eta(\tau)}\theta_{0,2}(\tau,-\rho)X^{sl(2/1);R}_{\frac{1}{2},0;0}(\tau,\sigma,\nu) \tag{C.0.5}$$

Now we want to compare it with our generalised formula as we told before, that $s\hat{l}(2/1)$ characters are periodic when $\theta \rightarrow \beta + un$, $n \in \mathbb{Z}$, and $p = 1$ where $\beta = 0, ...u-1$. The RHS of the (4.3) in this case becomes:

$$\sum_{\beta=0}^{1}\sum_{n\in\mathbb{Z}} X^{s\hat{l}(2/1);R}_{\frac{2-s'}{2}-\frac{1}{4}(s-1),\frac{2-s'}{2}-\frac{1}{4}(s+1),-\frac{1}{2};\beta}(\tau,\sigma,\nu)q^{\frac{1}{8}(3-s-2s'-2(2n+\beta))^2}\frac{y^{6s'-s-4n-2\beta}}{\prod_{i=1}^{n}(1-q^i)},$$

where $s' = 1$, so we have

$$\sum_{\beta=0}^{1} X^{s\hat{l}(2/1);R}_{\frac{3-s}{4},-\frac{1}{2};\beta}(\tau,\sigma,\nu)\sum_{n\in\mathbb{Z}} q^{(n+\frac{2\beta+s-1}{4})^2}y^{-2(n+\frac{2\beta+s-1}{4})} =$$
$$\sum_{\beta=0}^{1} X^{s\hat{l}(2/1);R}_{\frac{3-s}{4},-\frac{1}{2};\beta}(\tau,\sigma,\nu)\frac{\theta_{2\beta+s-1,2}(\tau,-\rho)}{\eta(\tau)}. \tag{C.0.6}$$

Let us now, collect all the information, we have two cases, first of all we shall mainly consider $s = 1$ and $s' = 1$. Finally we will arrive at

$$\frac{1}{\eta(\tau)}\theta_{0,2}(\tau,-\rho)X^{s\hat{l}(2/1);R}_{\frac{1}{2},0,-\frac{1}{2};0}(\tau,\sigma,\nu)+\frac{1}{\eta(\tau)}\theta_{2,2}(\tau,-\rho)X^{s\hat{l}(2/1);R}_{\frac{1}{2},0,-\frac{1}{2};1}(\tau,\sigma,\nu)=$$
$$X^{sl(2)}_{3,2,1,1}(\tau,\sigma)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}-\rho)+$$
$$X^{sl(2)}_{3,2,2,1}(\tau,\sigma)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}-\rho) \tag{C.0.7}$$

This is totally aggress with the sumrule $\lambda = 0$ in [66]. For s=2 and s'=1 we can write as;

$$\begin{array}{l}\frac{1}{\eta(\tau)}\theta_{1,2}(\tau,-\rho)X^{s\hat{l}(2/1);R}_{\frac{1}{4},-\frac{1}{4},-\frac{1}{2};0}(\tau,\sigma,\nu)+\frac{1}{\eta(\tau)}\theta_{3,2}(\tau,-\rho)X^{s\hat{l}(2/1);R}_{\frac{1}{4},-\frac{1}{4},-\frac{1}{2};1}(\tau,\sigma,\nu)=\\ X^{sl(2)}_{3,2,1,2}(\tau,\sigma)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}-\rho)+\\ X^{sl(2)}_{3,2,2,2}(\tau,\sigma)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}-\rho).\end{array} \tag{C.0.8}$$

This totally agrees with the sumrule $\lambda = 1$ in [66], as can be checked in the following.

$$\begin{aligned} B^R_1 &= \theta_{2,1}(\tau,\frac{\nu}{2}-\rho)X^{sl(2)}_{3,2,1,2}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}-\rho)+\\ &\quad \theta_{1,1}(\tau,\frac{\nu}{2}-\rho)X^{sl(2)}_{3,2,2,2}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}-\rho)\end{aligned} \tag{C.0.9}$$

and

$$A^R_1 = \theta_{3,2}(\tau,-\rho)X^{s\hat{l}(2/1);IV}_{1,0}(\tau,\sigma,\rho)+\theta_{1,2}(\tau,-\rho)X^{s\hat{l}(2/1);IV}_{1,1}(\tau,\sigma,\rho) \tag{C.0.10}$$

The same as before, we can write as

$$\begin{array}{l}\frac{1}{\eta(\tau)}\theta_{2,1}(\tau,\frac{\nu}{2}-\rho)=X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}-\rho)\\ \frac{1}{\eta(\tau)}\theta_{,1}(\tau,\frac{\nu}{2}-\rho)=X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}-\rho).\end{array} \tag{C.0.11}$$

also we can write for class IV character,

$$\begin{array}{l}X^{s\hat{l}(2/1);IV}_{1,0}(\tau,\sigma,\nu)=X^{s\hat{l}(2/1);R}_{\frac{1}{4},-\frac{1}{4},-\frac{1}{2};1}(\tau,\sigma,\nu)\\ X^{s\hat{l}(2/1);IV}_{1,1}(\tau,\sigma,\nu)=X^{s\hat{l}(2/1);R}_{\frac{1}{4},-\frac{1}{4},-\frac{1}{2};0}(\tau,\sigma,\nu)\end{array} \tag{C.0.12}$$

The sumrule for this case is :

$$\begin{array}{l}X^{sl(2)}_{3,2,1,2}(\tau,\sigma)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}-\rho)+\\ X^{sl(2)}_{3,2,2,2}(\tau,\sigma)X^{sl(2)}_{3,1,1,1}(\tau,\frac{\nu}{2}+\rho)X^{sl(2)}_{3,1,2,1}(\tau,\frac{\nu}{2}-\rho)=\\ \frac{1}{\eta(\tau)}\theta_{3,2}(\tau,-\rho)X^{s\hat{l}(2/1);R}_{\frac{1}{4},-\frac{1}{4},-\frac{1}{2};1}(\tau,\sigma,\nu)+\frac{1}{\eta(\tau)}\theta_{1,2}(\tau,-\rho)X^{s\hat{l}(2/1);R}_{\frac{1}{4},-\frac{1}{4},-\frac{1}{2};0}(\tau,\sigma,\nu)\end{array} \tag{C.0.13}$$

Appendix D

Discussion about p+u even and odd

Our starting point is (4.33), where we use

$$\sin\frac{\pi p(p+u-r)r'''}{p+u} = -e^{i\pi p r'''}\sin\frac{\pi p r r'''}{p+u}, \tag{D.0.1}$$

as well as the exponential representation of sinus to write,

$$\begin{aligned}
\sigma(\nu,\mu,\rho,\tau) = (\frac{1}{2i})^2 \sum_{r=1}^{p+u-1}\sum_{r''=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{r'''=1}^{p+u-1}\sum_{s'''=1}^{p}\sum_{r^{iv}=1}^{2} \\
e^{i\pi\left[(s-1)(r''-\frac{p+u}{u}(s''-1))+(s'-1)(r'''-\frac{p+u}{p}(s'''-1))+(p+u-r)(s'''-1)+r(s''-1)+pr'''\right]} \\
\times\left[e^{\frac{i\pi r}{p+u}(ur''-pr''')}+e^{-\frac{i\pi r}{p+u}(ur''-pr''')}-e^{\frac{i\pi r}{p+u}(ur''+pr''')}-e^{-\frac{i\pi r}{p+u}(ur''+pr''')}\right] \\
\times\sin\frac{\pi(2-\epsilon(p+1+s+s'+r)r^{iv}}{3} \\
\times X^{\hat{sl}(2)}_{r'',s'',p+u,u}(\nu,\tau)X^{\hat{sl}(2)}_{r''',s''',p+u,p}(\frac{p}{u}\mu+\rho,\tau)X^{\hat{sl}(2)}_{r^{iv},1,3,1}(\frac{u-p}{u}\mu-\rho,\tau).
\end{aligned} \tag{D.0.2}$$

Our analysis is divided in two cases: $p+u$ even and odd. We note that, for r even,

$$\sin\frac{\pi(2-\epsilon(p+1+s+s'+r)r^{iv}}{3} = \frac{\sqrt{3}}{2}\left[\epsilon(p+1+s+s')+[1-\epsilon(p+1+s+s')]e^{i\pi(1+r^{iv})}\right], \tag{D.0.3}$$

while for r odd,

$$\sin\frac{\pi(2-\epsilon(p+1+s+s'+r)r^{iv}}{3} = \frac{\sqrt{3}}{2}\left[\epsilon(p+s+s')+[1-\epsilon(p+s+s')]e^{i\pi(1+r^{iv})}\right], \tag{D.0.4}$$

so that we split our study of each parity of $p+u$ according to the parity of r.

Here, we will only detail the case where $p+u$ is even, but only because the treatment of th ecase $p+u$ odd is similar and would make this appendix tedious. The result for the case $p+u$ odd is quoted in (D.0.26).

$p+u$ **even**. Note that since p and u are coprime, they must both be odd in this case.

- r even We relabel $r=2k$, so that in this case, (D.0.2) becomes,

$$\sigma(\nu,\mu,\rho,\tau) = \frac{\sqrt{3}}{2}(\frac{1}{2i})^2 \sum_{r''=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{r'''=1}^{p+u-1}\sum_{s'''=1}^{p}\sum_{r^{iv}=1}^{2}$$
$$e^{i\pi\left[pr'''+(s-1)(r''-\frac{p+u}{u}(s''-1))+(s'-1)(r'''-\frac{p+u}{p}(s'''-1))\right]}$$
$$\times\left[A+A^*-C-C^*\right]\left[\epsilon(p+1+s+s')-[1-\epsilon(p+1+s+s')]e^{i\pi r^{iv}}\right]$$
$$\times X^{\hat{sl}(2)}_{r'',s'',p+u,u}(\nu,\tau)X^{\hat{sl}(2)}_{r''',s''',p+u,p}(\frac{p}{u}\mu+\rho,\tau)X^{\hat{sl}(2)}_{r^{iv},1,3,1}(\frac{u-p}{u}\mu-\rho,\tau), \quad \text{(D.0.5)}$$

where

$$A = \sum_{k=1}^{\frac{1}{2}(p+u-2)} e^{\frac{2i\pi k}{p+u}(ur''-pr''')}$$
$$C = \sum_{k=1}^{\frac{1}{2}(p+u-2)} e^{\frac{2i\pi k}{p+u}(ur''+pr''')}. \quad \text{(D.0.6)}$$

If $ur''-pr''' = \alpha(p+u)$, for $\alpha\in\mathbb{Z}$, i.e. if $(r'',r''')=(p+u-j,j)$ for $j=1,...,p+u-1$, then $A+A^*=p+u-2$. If $ur''-pr'''$ is not proportional to $(p+u)$ and is even, use

$$\sum_{\alpha=1}^{N} e^{\alpha\Gamma} = e^{\Gamma}\frac{1-e^{N\Gamma}}{1-e^{\Gamma}} \quad \text{(D.0.7)}$$

with $N=\frac{1}{2}(p+u-2)$ and $\Gamma=\frac{2i\pi(ur''-pr''')}{p+u}$ to conclude $A+A^*=-2$. If $ur''-pr'''$ is not proportional to $(p+u)$ and is odd, $A+A^*=0$. Similarly, if $ur''+pr'''=\alpha(p+u)$, for $\alpha\in\mathbb{Z}$, i.e. if $(r'',r''')=(p+u-j,j)$ for $j=1,...,p+u-1$, then $C+C^*=p+u$; if $ur''+pr'''$ is not proportional to $(p+u)$ and is even,$C+C^*=-2$ and if $ur''+pr'''$ is not proportional to $(p+u)$ and is odd, $C+C^*=0$.

So the non-zero contributions to $A + A^* - C - C^*$ come from two sources. When $(r'', r''') = (p+u-j, j)$ with $j = 1, ..., p+u-1$, $A+A^* = p+u-2$ and $C + C^* = -2$ because $ur'' + pr''' = u(p+u-j) + pj = u(p+u) + j(p-u)$ is always even. *However*, when $j = \frac{1}{2}(p+u)$, $C + C^* = p+u-2$. In conclusion, $A + A^* - C - C^* = p+u$ when $(r'', r''') = (p+u-j, j)$ with $j = 1, ..., p+u-1$, except $j = \frac{1}{2}(p+u)$. Similarly, $A + A^* - C - C^* = -(p+u)$ when $(r'', r''') = (j, j)$ with $j = 1, ..., p+u-1$, except $j = \frac{1}{2}(p+u)$.

We write,

$$\begin{aligned}\sigma_{even}(\nu, \mu, \rho, \tau) &= \frac{\sqrt{3}}{8}(p+u) \\ &\times \Big\{ \sum_{r^{iv}=1}^{2} [\epsilon(p+s+s') e^{i\pi r^{iv}} - (1 - \epsilon(p+s+s'))] \Big\} X^{\hat{sl}(2)}_{r^{iv},1,3,1}(\frac{u-p}{u}\mu - \rho, \tau) \\ &\quad \times \sum_{j=1}^{p+u-1} \sum_{s''=1}^{u} \sum_{s'''=1}^{p} e^{i\pi \left[pj + (s-1)(j - \frac{p+u}{u}(s''-1)) + (s'-1)(j - \frac{p+u}{p}(s'''-1))\right]} \\ &\times \Big\{ X^{\hat{sl}(2)}_{p+u-j,s'',p+u,u}(\nu,\tau) X^{\hat{sl}(2)}_{j,s''',p+u,p}(\frac{p}{u}\mu + \rho, \tau) - X^{\hat{sl}(2)}_{j,s'',p+u,u}(\nu,\tau) X^{\hat{sl}(2)}_{j,s''',p+u,p}(\frac{p}{u}\mu + \rho, \tau) \Big\}\end{aligned} \tag{D.0.8}$$

- r odd

We relabel $r = 2k - 1$, so that in this case, (D.0.2) becomes,

$$\begin{aligned}\sigma(\nu, \mu, \rho, \tau) &= \frac{\sqrt{3}}{2} (\frac{1}{2i})^2 \sum_{r''=1}^{p+u-1} \sum_{s''=1}^{u} \sum_{r'''=1}^{p+u-1} \sum_{s'''=1}^{p} \sum_{r^{iv}=1}^{2} \\ &\quad e^{i\pi \left[pr''' + s'' + s''' + (s-1)(r'' - \frac{p+u}{u}(s''-1)) + (s'-1)(r''' - \frac{p+u}{p}(s'''-1))\right]} \\ &\quad \times \left[\tilde{A} + \tilde{A}^* - \tilde{C} - \tilde{C}^*\right] \left[\epsilon(p+s+s') + [1 - \epsilon(p+s+s')] e^{i\pi(1+r^{iv})}\right] \\ &\times X^{\hat{sl}(2)}_{r'',s'',p+u,u}(\nu,\tau) X^{\hat{sl}(2)}_{r''',s''',p+u,p}(\frac{p}{u}\mu + \rho, \tau) X^{\hat{sl}(2)}_{r^{iv},1,3,1}(\frac{u-p}{u}\mu - \rho, \tau),\end{aligned} \tag{D.0.9}$$

where

$$\begin{aligned}\tilde{A} &= \sum_{k=1}^{\frac{1}{2}(p+u)} e^{\frac{i\pi(2k-1)}{p+u}(ur'' -- pr''')} \\ \tilde{C} &= \sum_{k=1}^{\frac{1}{2}(p+u)} e^{\frac{i\pi(2k-1)}{p+u}(ur'' + pr''')}.\end{aligned} \tag{D.0.10}$$

The non-zero contributions to $\tilde{A} + \tilde{A}^* - \tilde{C} - \tilde{C}^*$ come from two sources. When $(r'', r''') = (p+u-j, j)$ with $j = 1, ..., p+u-1$, $\tilde{A} + \tilde{A}^* = (-1)^{u-j}(p+u) =$

$(-1)^{j+1}(p+u)$ and $\tilde{C}+\tilde{C}^*=0$ *except* when $j=\frac{1}{2}(p+u)$, when $\tilde{C}+\tilde{C}^*=(-1)^{\frac{1}{2}(p+u)}(p+u)$ so that $\tilde{A}+\tilde{A}^*-\tilde{C}-\tilde{C}^*=0$.

In conclusion, $\tilde{A}+\tilde{A}^*-\tilde{C}-\tilde{C}^*=(-1)^{j+1}(p+u)$ when $(r'',r''')=(p+u-j,j)$ with $j=1,...,p+u-1$, except $j=\frac{1}{2}(p+u)$. Similarly, $\tilde{A}+\tilde{A}^*-\tilde{C}-\tilde{C}^*=(-1)^{j+1}(p+u)$ when $(r'',r''')=(j,j)$ with $j=1,...,p+u-1$, except $j=\frac{1}{2}(p+u)$.

We write,

$$\begin{aligned}\sigma_{odd}(\nu,\mu,\rho,\tau) &= \frac{\sqrt{3}}{8}(p+u)\\ &\times\Big\{\sum_{r^{iv}=1}^{2}[(1-\epsilon(p+s+s'))e^{i\pi r^{iv}}-\epsilon(p+s+s')]\Big\}X^{\hat{sl}(2)}_{r^{iv},1,3,1}(\frac{u-p}{u}\mu-\rho,\tau)\\ &\times\sum_{j=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{s'''=1}^{p}e^{i\pi\left[1+s''+s'''+(s-1)(j-\frac{p+u}{u}(s''-1))+(s'-1)(j-\frac{p+u}{p}(s'''-1))\right]}\\ &\times\Big\{X^{\hat{sl}(2)}_{p+u-j,s'',p+u,u}(\nu,\tau)X^{\hat{sl}(2)}_{j,s''',p+u,p}(\frac{p}{u}\mu+\rho,\tau)+X^{\hat{sl}(2)}_{j,s'',p+u,u}(\nu,\tau)X^{\hat{sl}(2)}_{j,s''',p+u,p}(\frac{p}{u}\mu+\rho,\tau)\Big\}\end{aligned} \tag{D.0.11}$$

We now proceed to express $\sigma_{even}+\sigma_{odd}$ when $p+u$ is even. First note that,

$$\sum_{r^{iv}=1}^{2}[\epsilon(p+s+s')e^{i\pi r^{iv}}-(1-\epsilon(p+s+s'))]X^{\hat{sl}(2)}_{r^{iv},1,3,1}= -X^{\hat{sl}(2)}_{1,1,3,1}+e^{i\pi(s+s')}X^{\hat{sl}(2)}_{2,1,3,1}, \tag{D.0.12}$$

while

$$\sum_{r^{iv}=1}^{2}[(1-\epsilon(p+s+s'))e^{i\pi r^{iv}}-\epsilon(p+s+s')]X^{\hat{sl}(2)}_{r^{iv},1,3,1}= -X^{\hat{sl}(2)}_{1,1,3,1}-e^{i\pi(s+s')}X^{\hat{sl}(2)}_{2,1,3,1}. \tag{D.0.13}$$

Inserting these expressions in (D.0.8) and (D.0.11), and after some elementary steps, we write, omitting the character arguments for clarity,

$$\sigma = \sigma_{even}+\sigma_{odd} = \frac{\sqrt{3}}{8}(p+u)\sum_{j=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{i\pi\left[(1+s+s')j-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s'''-1)\frac{p+u}{p}\right]}$$

$$\times\Bigg\{ \left\{[e^{i\pi(s''+s'''+j)}-1]X^{\hat{sl}(2)}_{1,1,3,1}+e^{i\pi(s+s')}[1+e^{i\pi(s''+s'''+j)}]X^{\hat{sl}(2)}_{2,1,3,1}\right\}X^{\hat{sl}(2)}_{j,s'',p+u,u}X^{\hat{sl}(2)}_{p+u-j,s''',p+u,p}$$

$$+\left\{[e^{i\pi(s''+s'''+j)}+1]X^{\hat{sl}(2)}_{1,1,3,1}-e^{i\pi(s+s')}[1-e^{i\pi(s''+s'''+j)}]X^{\hat{sl}(2)}_{2,1,3,1}\right\}X^{\hat{sl}(2)}_{j,s'',p+u,u}X^{\hat{sl}(2)}_{j,s''',p+u,p}\Bigg\} \tag{D.0.14}$$

The first two terms in (D.0.14) may be rewritten as,

$$\frac{\sqrt{3}}{4}(p+u)\sum_{j=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{i\pi\left[1+(1+s+s')j-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s'''-1)\frac{p+u}{p}\right]}$$
$$\times\, e^{i\pi(1+s+s')\epsilon(s''+s'''+j+1)}X^{\hat{sl}(2)}_{j,s'',p+u,u}X^{\hat{sl}(2)}_{p+u-j,s''',p+u,p}X^{\hat{sl}(2)}_{2-\epsilon(s''+s'''+j),1,3,1}, \tag{D.0.15}$$

but

$$e^{i\pi(1+s+s')(j+\epsilon(s''+s'''+j+1))} = e^{i\pi(1+s+s')(1+s''+s''')}, \tag{D.0.16}$$

so that these terms give,

$$\frac{\sqrt{3}}{4}(p+u)\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{i\pi\left[1-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s'''-1))\frac{p+u}{p}\right]}\times e^{i\pi(1+s+s')(1+s''+s''')}$$
$$\sum_{j=1}^{p+u-1} X^{\hat{sl}(2)}_{j,s'',p+u,u}X^{\hat{sl}(2)}_{p+u-j,s''',p+u,p}X^{\hat{sl}(2)}_{2-\epsilon(s''+s'''+j),1,3,1}. \tag{D.0.17}$$

Using the Neveu-Schwarz sumrules (4.16), and remembering that here, p is odd, and therefore $\epsilon(s''+s'''+j)=\epsilon(s''+s'''+j+p+1)$, we have,

$$\sum_{j=1}^{p+u-1} X^{\hat{sl}(2)}_{j,s'',p+u,u}X^{\hat{sl}(2)}_{p+u-j,s''',p+u,p}X^{\hat{sl}(2)}_{2-\epsilon(s''+s'''+j),1,3,1} =$$
$$q^{\frac{p}{4u}}y^{-\frac{1}{2}}\sum_{\theta\in\mathbb{Z}} X^{\hat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'',u,p;\theta+\frac{1}{2}}\frac{y^{A(s'',s''',\theta)}q^{\frac{p}{u}[A^2(s'',s''',\theta)-A(s'',s''',\theta)]}}{\eta(q)}. \tag{D.0.18}$$

The first two terms in (D.0.14) become,

$$\frac{\sqrt{3}}{4}(p+u)q^{\frac{p}{4u}}y^{-\frac{1}{2}}\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{i\pi\left[1+(1+s+s')(1+s''+s''')-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s'''-1)\frac{p+u}{p}\right]}$$
$$\sum_{\theta\in\mathbb{Z}} X^{\hat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'',u,p;\theta+\frac{1}{2}}\frac{y^{A(s'',s''',\theta)}q^{\frac{p}{u}[A^2(s'',s''',\theta)-A(s'',s''',\theta)]}}{\eta(q)}. \tag{D.0.19}$$

he last two terms in (D.0.14) may be rewritten as,

$$-\frac{\sqrt{3}}{4}(p+u)\sum_{j=1}^{p+u-1}\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{i\pi\left[1+(1+s+s')j-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s'''-1)\frac{p+u}{p}\right]}$$
$$\times e^{i\pi(1+s+s')\epsilon(s''+s'''+j)}X^{\hat{sl}(2)}_{j,s'',p+u,u}X^{\hat{sl}(2)}_{p+u-j,s''',p+u,p}X^{\hat{sl}(2)}_{2-\epsilon(s''+s'''+j+1),1,3,1}, \quad \text{(D.0.20)}$$

or again, using (4.16) and

$$e^{i\pi(1+s+s')(j+\epsilon(s''+s'''+j))} = e^{i\pi(1+s+s')(s''+s''')}, \quad \text{(D.0.21)}$$

one has,

$$\frac{\sqrt{3}}{4}(p+u)\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{i\pi\left[1+(1+s+s')(s''+s''')-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s'''-1)\frac{p+u}{p}\right]}$$
$$q^{\frac{p}{4u}(u+1)^2}y^{-\frac{1}{2}(u+1)}\sum_{\theta\in\mathbb{Z}}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s''+u,u,p;\theta+\frac{1}{2}}\frac{y^{A(s'',s''',\theta)}q^{\frac{p}{u}[A^2(s'',s''',\theta)-A(s'',s''',\theta)(u+1)]}}{\eta(q)}.$$

(D.0.22)

Now use the $\widehat{sl}(2\mid 1;\mathbb{C})$ property (3.54), i.e.

$$X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s''+u,u,p;\theta+\frac{1}{2}} = X^{\widehat{sl}(2|1;\mathbb{C})}_{1-s''',s'',u,p;\theta+\frac{1}{2}}, \quad \text{(D.0.23)}$$

and relabel $s'''=s^{iv}-p$, so that (D.0.22) is given by,

$$\frac{\sqrt{3}}{4}(p+u)\sum_{s''=1}^{u}\sum_{s^{iv}=1+p}^{2p} e^{i\pi\left[1+(1+s+s')(s''+s^{iv}+1)-(s-1)(s''-1)\frac{p+u}{u}-(s'-1)(s^{iv}-1)\frac{p+u}{p}\right]}$$
$$q^{\frac{p}{4u}}y^{-\frac{1}{2}}\sum_{\theta\in\mathbb{Z}}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s^{iv},s'',u,p;\theta+\frac{1}{2}}\frac{y^{A(s'',s^{iv},\theta)}q^{\frac{p}{u}[A^2(s'',s^{iv},\theta)-A(s'',s^{iv},\theta)]}}{\eta(q)}. \quad \text{(D.0.24)}$$

By combining the above contribution (for r odd) with the contribution (D.0.19) (for r even), we write, *when $p+u$ is even,*

$$\sigma(\nu,\mu,\rho,\tau) = \frac{\sqrt{3}}{4}(p+u)\frac{1}{\eta(\tau)}\sum_{s''=1}^{u}\sum_{s'''=1}^{2p} e^{i\pi[ss'''+s's''-(s-1)(s''-1)\frac{p}{u}-(s'-1)(s'''-1)\frac{u}{p}]}$$
$$\times\sum_{\theta\in\mathbb{Z}}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'',u,p;\theta+\frac{1}{2}}(\nu,\mu,\tau)y^{A(s'',s''',\theta)-\frac{1}{2}}q^{\frac{p}{u}[A(s'',s''',\theta)-\frac{1}{2}]^2}. \quad \text{(D.0.25)}$$

When $p+u$ is odd, the expression (D.0.2) can be shown to give,

$$\sigma(\nu,\mu,\rho,\tau) = \frac{\sqrt{3}}{4}(p+u)\frac{1}{\eta(\tau)}\sum_{s''=1}^{u}\sum_{s'''=1}^{p} e^{\pi i(p+s+s')(s''+s''')}e^{\pi i[-(s-1)\frac{p+u}{u}(s''-1)-(s'-1)\frac{p+u}{p}(s'''-1)]}$$
$$\times e^{\pi i s'''}y^{-\frac{1}{2}(u+1)}q^{\frac{p}{4u}+\frac{p}{2}+\frac{up}{4}}\sum_{\theta\in\mathbb{Z}}X^{\widehat{sl}(2|1;\mathbb{C})}_{p+1-s''',s'+u',u,p;\theta}(\nu,\mu,\tau)y^{A(s'',s''',\theta)}q^{\frac{p}{u}[A(s'',s''',\theta)^2-A(s'',s''',\theta)]}.$$

(D.0.26)

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